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MESSENGER OF MATHEMATICS.

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MESSENGER OF MATHEMATICS.

ON THE ORDER OF ORTHOPTIC LOCI.

By *C. Taylor, D.D.*

If to any curve there be drawn a pair of tangents at right angles to one another, the locus of their point of concourse may be called its *Orthoptic Locus,* since at every point thereof the curve subtends or is seen under a right angle.

The first step towards a theory of such loci was the determination of the orthoptic locus of a conic. This was shewn by De la Hire to be a circle, which in the case of the parabola degenerates into a straight line—or rather into a line-circle, consisting of the directrix and the line at infinity. One of the ways in which this result may be arrived at is by finding in what points the locus meets the line at infinity; and the same method will be seen to be applicable to curves of all classes. The proof depends upon a peculiarity of the two imaginary points at infinity through which all circles in a plane pass.

1. *The circular points at infinity.*

Straight lines which meet at infinity are in all cases parallel, and in general cannot be regarded as inclined at any finite angle. But an exception must be made in the case of the two imaginary points I and J at infinity through which all circles pass.

(a) For through one of them I draw any two lines IA and IB. Then the points A and B subtend any assumed angle α at all points on the circumference of a certain circle, and therefore at I. Therefore IA and IB, although they meet at infinity, may nevertheless be regarded as including the angle α, which may be of any magnitude whatsoever. That is to say, the angle between any two lines drawn through one of the circular points is *indeterminate*, and may be said to be of any assumed magnitude.

Hence, by supposing the points A and B to coincide, we see that any straight line IA drawn through a circular point makes *an indeterminate angle with itself*.

It is commonly said that such a line is at right angles to itself. But the statement should not be so limited. For the tangent of the angle between the lines $y - mx = 0$, and $y - m'x = 0$, is $\frac{m - m'}{1 + mm'}$, so that when $m = m' = \sqrt{-1}$, not only does the denominator vanish, but the numerator also. The angle which any line $y = \sqrt{-1} . x + c$ makes with itself is therefore indeterminate.

(*b*) The existence of the circular points may be proved as follows.

On a given circle take points A and B. These subtend an angle α at the circumference, and therefore at either of the points in which the circle meets the line at infinity. Call these I and J, and draw IA' and IB' to any points A' and B' in the plane.

Then since I is at infinity, the angle AIA' is equal to zero,* and likewise the angle BIB'. Therefore the angle $A'IB'$ is equal to AIB, or α, which may be of any magnitude, according to the assumed positions of the points A and B on the circle.

That is to say, any two points A' and B' in the plane subtend an indeterminate angle at I; and therefore all circles in the plane pass through I, and in like manner through J; and no finite circle can pass through more than these two points on the line at infinity.

1. *The ellipse.*

If two tangents be drawn from I or J to any curve, they may be said to be at right angles. These two points therefore belong to the orthoptic locus of the curve; and in general no other points at infinity belong to it.

(*a*) The ellipse being a curve of the second class, two tangents only can be drawn to it from either of the circular points. These are therefore *single* points on the orthoptic locus, and the locus is accordingly a circle.

At the points in which this locus cuts the ellipse the tangent to the ellipse must be at right angles to itself, and and must therefore pass through I or J. That is to say, it must be a tangent from one of the foci, and must therefore have its point of contact on the corresponding directrix, the polar of the focus. The orthoptic locus is therefore *the circle through the points in which the ellipse cuts its directrices.*

* An angle subtended at a point on the line at infinity may always be said to be equal to zero, or to have zero for one of its values.

(b) Since SI and SJ are tangents, each at right angles to itself, these two lines, which make up the point-circle at the focus S, might be said to belong, in a special sense, to the orthoptic locus. But in this theory the degree of the locus is estimated independently of such factors.

If two tangents at right angles have for their equations,

$$y - mx = \sqrt{(b^2 + m^2a^2)},$$

and

$$my + x = \sqrt{(m^2b^2 + a^2)},$$

the equation to their locus of intersection will be,

$$(1 + m^2)(x^2 + y^2 - a^2 - b^2) = 0,$$

where the $(1 + m^2)$ points to the tangents from the foci.

2. *The cardioid.*

It is known that in the cardioid, according as the chord of contact of the orthogonal tangents does or does not pass through the cusp, their point of concourse traces a *circle* or a *bicircular quartic.*

These together make up a tricircular sextic; and such should be the orthoptic locus of any curve of the third class. For from I or J three tangents can be drawn to it; and these combine to make three quasi-orthogonal pairs of tangents. Each of the circular points is therefore a threefold point on the orthoptic locus, and these are in general its only points at infinity.

3. *Of orthoptic loci in general.*

To a curve of the n^{th} class n tangents can be drawn from I or J; and these may be taken in pairs in $n\,\frac{n-1}{2}$ ways; and every pair may be regarded as at right angles to one another.

Each of the circular points is therefore a point of the order $n\,\frac{n-1}{2}$ on the orthoptic locus; and they are in general its only points at infinity. Consequently the order or degree of the locus is $n(n-1)$. That is to say, *the orthoptic locus of any curve of class n is of the order $n(n-1)$, and it passes $n\,\frac{n-1}{2}$ times through the circular points.*

In verification of this result, notice that when the curve of class n degenerates into n points, its orthoptic locus evidently consists of the $n\,\frac{n-1}{2}$ circles which have the lines joining the points two and two for diameters. In this we omit the point-circles at the n points, which might have been regarded as factors of the locus.

4. *The parabola.*

Of the two tangents that can be drawn to a parabola from any point on the line at infinity one is that line itself, which may be regarded as orthogonal to every other tangent.

Every point on the line at infinity is therefore a single point on the orthoptic locus of the parabola; and the remaining factor of the locus must be a straight line not at infinity.

This straight line is the directrix, the polar of the focus; since each of the tangents to the parabola from the focus is at right angles to itself, and therefore has its point of contact on the orthoptic locus.

Conversely, given that the directrix is an orthoptic locus, the parabola must touch the line at infinity; for the tangents to it from the point at infinity on the directrix could only be regarded as parallel unless one of them were the line at infinity.

There is a corresponding reduction in the order of the orthoptic locus of a curve which has contact at more points than one with the line at infinity.

5. *The order of pedals.*

Given a curve of the n^{th} class and a point O, the locus of the foot of the perpendicular from O to a tangent to the curve will be an n-circular curve of the order $2n$.

For the line OI or OJ may be regarded as perpendicular to every one of the n tangents from I or J respectively. Each of the circular points is therefore an n-fold point on the pedal, and this (having no other points at infinity) is of the order $2n$.

When O coincides with a focus, each of the lines OI, OJ is a tangent, and is also perpendicular to itself. Consequently these lines, which make up the point-circle at O, belong to the locus; and its remaining factor is of the order $2(n-1)$.

For example, in the ellipse, in which any tangent and the perpendiculars upon it from a focus may be represented by the equations,

$$y - mx = \sqrt{(b^2 + m^2a^2)},$$

and

$$my + x = \sqrt{(a^2 - b^2)},$$

the equation to the pedal with respect to a focus is,

$$(1 + m^2)(x^2 + y^2 - a^2) = 0.$$

The pedal therefore (which is in general a bicircular