MATHEMATICAL QUESTIONS WITH THEIR SOLUTIONS. FROM THE "EDUCATIONAL TIMES"

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W. J. MILLER

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EDITED BY

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(1) $a_1^2+a_2^2+a_3^2=\frac{1}{2}(a^2+b^2+c^2)$.

(2) $b_1^2=b_2^2=b_3^2=4R^2\cos A\cos B\cos C$.

(3) $h_1^2+h_2^2+h_3^2=4R^2\cos A\cos B\cos C$.

(4) $e_1^2+e_2^2+e_3^2=2$.

(5) The portions of the sides intercepted by the ellipses are semi-conjugate diameters.

(6). The tangents at the points where the ellipses cut the perpendiculars meet the sides in points which lie on lines parallel to the sides of the triangle.

(7) The common chords of the ellipses pass through the angles of the triangle........ 28

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$$\frac{dy\,ds^2}{dx\,d^2s}=a\ constant\ \ldots\ldots\ (1).$$

$$\frac{dy\,d^3y}{(d^2s)^2}=\frac{ds^2\,d^3y}{dy\,(d^2y)^2}=a\ constant\ \ldots\ldots\ (2).$$

$$\frac{d^2y}{dy}+\frac{d^4y}{d^3y}=2\frac{d^3s}{d^2s}\ \ldots\ldots\ (3).$$

$$\frac{d^2y}{dy}+2\frac{d^3y}{d^2y}-\frac{d^4y}{d^3y}=2\frac{d^2s}{ds}\ \ldots\ldots\ (4).$$ 34

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$$\frac{(CP')^2}{RP\,.\,Rp}+\frac{(CQ')^2}{rQ\,.\,rq}=\frac{(CS)^2}{(CB)^2}$$ 62

1428. From a point in the base of a triangle two straight lines are drawn containing a given

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angle, and forming a quadrilateral with the other two sides of the triangle. It is required to prove that

(α) Of all the quadrilaterals which have a common vertex at the *same* point within the vertical angle of the triangle, that of which the sides passing through this point are equal to each other is a *maximum*, a *minimum*, or *neither* a maximum nor a minimum, according as the given angle is *less* than, *greater* than, or *equal* to the supplement of the vertical angle of the triangle.

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1443. Show that the locus of the centres of all the conics circumscribing a given quadrilateral is an ellipse if the quadrilateral is re-entrant, and an hyperbola if it is convex. Show further that two real parabolas may always be drawn through the angles of any convex quadrilateral 51

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