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A unified method to analyze overtake free queueing systems

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Abstract

In this paper we demonstrate that the distributional laws that relate the number of customers in the system (queue), L (Q) and the time a customer spends in the system (queue), S (W) under the first-in-first-out (FIFO) discipline lead to a complete solution for *the distributions* of L, Q, S, W for queueing systems which satisfy distributional laws for both L and Q (*overtake free systems*). Moreover, in such systems the derivation of the distributions of L, Q, S, W can be done in *a unified way*. Our results include a generalization of PASTA to queueing systems with arbitrary renewal arrivals under heavy traffic conditions, a generalization of the Pollaczek-Khinchin formula to the $GI/G/1$ queue, an extension of the Fuhrmann and Cooper decomposition for queues with generalized vacations under mixed generalized Erlang renewal arrivals, new approximate results for the distributions of L, S in a $GI/G/\infty$ queue, and new exact results for the distributions of L, Q, S, W in priority queues with mixed generalized Erlang renewal arrivals.

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1 Introduction

What are the laws of electrodynamics? In order to address this question we should first define the fundamental quantities of electrodynamics, the electric field $\vec{E}$ and the magnetic field $\vec{B}$. The fundamental laws of electrodynamics are the Maxwell equations. The goal of electrodynamics is then to find $\vec{E}$ and $\vec{B}$ in various applications. The Maxwell equations form a *complete set* of laws in the sense that *just starting* from them and using the calculus of partial differential equations one is able to compute $\vec{E}$ and $\vec{B}$ either analytically or numerically in a variety of applications. What is important here is that the physics of a problem is summarized in the Maxwell equations, which then lead to a complete solution for $\vec{E}$ and $\vec{B}$ in a *unified way*.

Let us then ask the key question which motivated the present paper. What are the laws of queueing theory? The fundamental quantities in queueing theory are the stationary queue and system length $(Q,\ L)$ and the waiting and system time $(W,\ S)$ under the First-In-First-Out (FIFO) discipline. Of course there are several other random variables of interest (often particular to the application studied), but these are the most widely used. The goal of queueing theory is then to find the distributions of $Q,\ L,\ W,\ S$ in various applications. In its almost a hundred year history queueing theory has addressed a great variety of problems using a variety of techniques, which solve some problems but fail on others. What is interesting is the lack of *a unified way* to solve a particular application. Queueing theory research does not start from a set of well established laws and then proceed to the solution using some well established mathematical techniques. It rather uses the particular characteristics of the application to achieve its solution.

Coming to our original question regarding the laws of queueing theory, one would like to have a set of laws which, similar to Maxwell equations in electrodynamics, lead to a *complete* solution of the queueing application. One first candidate for a queueing law is Little's law [13] (see the recent review of Whitt [16] which traces the different forms of the law and its extensions). Let us examine whether Little's law leads to complete solution for the steady state $E[Q]$, $E[L]$, $E[W]$, $E[S]$ in a $GI/G/s$ queue. Let $\lambda,\ \mu,\ \rho = \frac{\lambda}{s\mu} < 1$ be the mean arrival, service rate and traffic intensity. Then, from Little's law in the system and the queue

$$E[L] = \lambda E[S], \; E[Q] = \lambda E[W].$$

But, $E[S] = E[W] + \frac{1}{\mu}$, while the relation of $Q, \; L$ is

$$E[z^L] = z^s E[z^Q] + \sum_{n=0}^{s-1} P\{L = n\}[z^n - z^s],$$

from where

$$E[L] = s + E[Q] - \sum_{n=0}^{s-1} (s - n) P\{L = n\}.$$

Combining the previous equations we obtain that

$$\sum_{n=0}^{s-1} \frac{s-n}{s} P\{L = n\} = 1 - \rho,$$

which is exactly what Little's law would give if it were applied to a service box including the customers in service. For example, in a $GI/G/1$ queue one would be able to find that $P\{L = 0\} = 1 - \rho$, but it would not be possible to find $E[L]$. As a result, despite its importance, Little's law does not lead to a complete solution for expected performance measures.

Our goal in this paper is to demonstrate that *the distributional laws* first obtained by Haji and Newell [7] are the fundamental queueing laws for queueing systems which satisfy distributional laws for both the number in the system and the number in the queue (we will call them *overtake free systems*). We demonstrate that the distributional laws lead to a complete solution for the stationary distributions of $L, \; Q, \; S, \; W$ in overtake free systems. Moreover, in such systems the derivation of the distributions of $L, \; Q, \; S, \; W$ can be done in *a unified way.* In this way not only we obtain new simple derivations of known results providing new insights to old results, but we obtain several new results as well. We propose two methods of analysis An asymptotic (as $\rho \to 1$) method which applies to overtake free systems with arbitrary renewal arrivals and an exact method which applies to overtake free systems with mixed generalized Erlang arrivals.

For the case of Poisson arrivals Keilson and Servi [10], [11] found that the distributional laws have a very convenient form that can lead to complete solutions for some overtake free systems. For the case of mixed generalized Erlang renewal arrivals Bertsimas and Nakazato [1] gave another proof of the distributional laws that lead to a very convenient form of the law. They also proposed a framework to find $E[L]$, $E[Q]$, $E[S]$, $E[W]$ in heavy traffic for overtake free queueing systems based on the distributional laws. In this paper we develop a methodology to find the distributions of L, Q, S, W for overtake free systems with arbitrary renewal arrivals, thus generalizing all earlier work. Our approach is to use asymptotic analysis (which is exact in heavy traffic) for the case of arbitrary renewal processes and exact analysis for the case of mixed generalized Erlang renewal arrivals.

The paper is structured as follows: In Section 2 we review the distributional laws. In Section 3 we present an asymptotic method of analysis for overtake free queueing systems based on the asymptotic properties of the distributional laws and a generalization of the well known result of Poisson arrivals see time averages (PASTA) to queueing systems with arbitrary renewal arrivals under heavy traffic conditions. Furthermore, we illustrate the efficiency of the method by deriving the distributions of L, Q, S, W in $GI/G/1$, $GI/D/s$ queues and obtaining new approximate results for the distributions of L, S in a $GI/G/\infty$ queue. Our derivation unifies the heavy traffic results and leads to a generalization of the Pollaczek-Khinchin formula to the $GI/G/1$ queue. In Section 4 we present an exact method of analysis for overtake free systems with mixed generalized Erlang (MGE) renewal arrivals and we implement it in the case of $MGE_M/G/1$ queue. This section demonstrates that there is a direct closed form expression for the number of customers in a $MGE_M/G/1$ system while our approach reproduces the known results for the waiting time involving roots of a certain nonlinear equation in a direct way without the need for Hilbert factorization. In Section 5, as another application of the exact method of analysis for overtake free systems, we extend the decomposition results for queues with generalized vacations considered in Fuhrmann and Cooper [5] for the $M/G/1$ queue to MGE arrivals.