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THE

# MESSENGER OF MATHEMATICS.

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MESSENGER OF MATHEMATICS.

# A NEW PROOF THAT A GENERAL QUADRIC MAY BE REDUCED TO ITS CANONICAL FORM (THAT IS, A LINEAR FUNCTION OF SQUARES) BY MEANS OF A REAL ORTHOGONAL SUBSTITUTION.

By Prof. *J. J. Sylvester*.

ALL the proofs that I am acquainted with (and their name is legion) of the possibility of depriving a quadric, in three or more variables, of its mixed terms by a real orthogonal transformation are made to depend on the theorem that the "*latent roots*" of any symmetrical matrix are all real.

By the latent roots is understood the roots of the determinant expressed by tacking on a variable $-\lambda$ to each term in the diagonal of symmetry to such matrix.

I shall show that the same conclusion may be established *à priori* by purely algebraical ratiocination and without constructing any equation, by the method of cumulative variation. The proof I employ is inductive: *i.e.* if the theorem is true for two or any number of variables I prove that it will be true for one more.

To illustrate the method let us begin with two variables. Consider the form $ax^2 + 2hxy + by^2$.

If in any such form $b = a$, then by an obvious orthogonal transformation, viz. writing $\frac{x+y}{\sqrt{(2)}}$ and $\frac{x-y}{\sqrt{(2)}}$ for $x$ and $y$, the the form becomes

$$a(x^2 + y^2) + h(x^2 - y^2),$$

or $$(a+h)x^2 + (a-h)y^2.$$

Now in general on imposing on $x, y$ any orthogonal infinitesimal substitution, so that

$$x \text{ becomes } x + \varepsilon y,$$
$$y \quad ,, \quad y - \varepsilon x$$

$h$ in the new form becomes $h + (a - b)\,\varepsilon$, or say $\delta h = (a - b)\,\varepsilon$, and $\frac{1}{2}\delta(h^2) = (a - b)\,h\varepsilon$; the variations of $a$ and $b$ need not be set forth.

Let an infinite succession of such transformations be instituted; then either $a$ and $b$ become equal and the orthogonal substitution above referred to reduces the quadric to its canonical form [in which case this one combined with the preceding infinite series of such substitutions may be compounded into a single substitution], or else by giving $\varepsilon$ the sign of $(b - a)$ the variation of $h^2$ may at each step be made negative so that $h^2$ continually decreases, unless $h$ vanishes. If $h$ does not vanish it must have a minimum value, and this minimum value may be diminished, which involves a contradiction: hence, in the infinite series of substitutions supposed, either $a$ and $b$ become equal or $h$ vanishes, and in either case the quadric is reduced or reducible to its canonical form.

Let us now take the case of three variables $x, y, z$.

Obviously, by the preceding case, we may make the term involving $xy$ disappear and commence with the initial form

$$ax^2 + by^2 + 2fxz + 2gyz + cz^2.$$

If $f$ or $g$ become zero the quadric may be canonified by virtue of the preceding case.

Again, if $b = a$ by imposing on $x, y$ the orthogonal substitution

$$\frac{g}{\sqrt{(f^2+g^2)}}\,x + \frac{f}{\sqrt{(f^2+g^2)}}\,y$$
$$-\frac{f}{\sqrt{(f^2+g^2)}}\,x + \frac{g}{\sqrt{(f^2+g^2)}}\,y,$$

the term involving $xz$ will disappear and the final result is the same as if $f$ were zero.

Let us now introduce the infinitesimal orthogonal substitution which changes

$$\begin{aligned} x &\text{ into } \quad x + \varepsilon y + \eta z, \\ y &\quad ,, \quad -\varepsilon x + y + \theta z, \\ z &\quad ,, \quad -\eta x - \theta y + z, \end{aligned}$$

where $\varepsilon, \eta, \theta$ are supposed to be of the same order of magnitude so that only first powers of them have to be considered.

Then
$$\delta f = (a-c)\,\eta - g\varepsilon,$$
$$\delta g = (b-c)\,\theta + f\varepsilon,$$
also the coefficient of $2xy$ becomes $(a-b)\,\varepsilon - f\theta - g\eta$.

Now whatever $\eta$, $\theta$ may be, we may determine $\varepsilon$ in terms of $\eta$, $\theta$ so that this may be made to vanish, and the initial form of the quadric will be maintained, provided that $b$ is not equal to $a$.

Hence instituting an infinite series of these infinitesimal substitutions, provided we do not reach a stage where $a$ and $b$ become equal, we may maintain the original form keeping $\eta$, $\theta$ arbitrary, and shall have
$$\tfrac{1}{2}\delta\,(f^2+g^2) = (a-c)\,f\eta + (b-c)\,g\theta.$$

Suppose $a$ and $b$ to be unequal; therefore $(a-c)$, $(b-c)$ do not vanish simultaneously, and consequently we may make $\delta\,(f^2+g^2)$ negative unless at least one of the two quantities $f$, $g$ vanishes.

If neither of them vanishes $f^2+g^2$ may be made continually to decrease and will have a minimum other than zero, which involves a contradiction.

Hence the infinite series of infinitesimal orthogonal substitutions may be so conducted that either $a-b$ or one at least of the letters $f$, $g$ shall become zero; and then two additional orthogonal substitutions at most will serve to reduce the Quadric immediately to its canonical form.

I shall go one step further to the case of four variables $x$, $y$, $z$, $t$, and then the course of the induction will become manifest. We may, by virtue of what has been shown, take as our quadric
$$ax^2 + by^2 + cz^2 + 2fxt + 2gyt + 2hzt + dt^2.$$

Here, if any one of the mixed terms disappears, the quadric is immediately reducible by the preceding case, and if any two of the grouped pure coefficients $a$, $b$, $c$ become equal (as for instance $a$, $b$), then by an orthogonal transformation one of the mixed terms ($f$ or $g$ in the case supposed) may be got rid of; so that this supposition merges in the preceding one.

Impose on $x, y, z, t$ an infinitesimal orthogonal substitution, writing
$$\begin{array}{ll}
x + \varepsilon y + \theta z + \lambda t & \text{for } x, \\
-\varepsilon x + y + \eta z + \mu t & \;\;,, \quad y, \\
-\theta x - \eta y + z + \nu t & \;\;,, \quad z, \\
-\lambda x - \mu y - \nu z + t & \;\;,, \quad t.
\end{array}$$