

**THE MESSENGER OF
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[MAY 1889 - APRIL 1890]**

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J. W. L. GLAISHER

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MESSENGER OF MATHEMATICS.

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CONTENTS.

ARITHMETIC AND ALGEBRA.		PAGE
A new proof that a general quadric may be reduced to its canonical form (i.e. a linear function of squares) by means of a real orthogonal substitution. By Prof. J. J. SYLVESTER	- - - -	1
Note on the sums of two series. By Prof. CAYLEY	- - - -	29
On the reduction of a bilinear quantic of the n th order to the form of a sum of n products by a double orthogonal substitution. By Prof. J. J. SYLVESTER	- - - -	43
On an arithmetical theorem in periodic continued fractions. By Prof. J. J. SYLVESTER	- - - -	63
Notes on modern higher algebra. By T. S. FUSKE	- - - -	89
Gordan's series. By H. F. BAKER	- - - -	91
Solution of $(a, b, \dots, c) = (a^p, b^p, \dots, c^p)$. By H. W. LLOYD TANNER	- - - -	118
On Latin squares. By Prof. CAYLEY	- - - -	135
Note on series whose coefficients involve powers of the Bernoullian numbers By J. W. L. GLAISHER	- - - -	138
A proof of the theorem of reciprocity for quadratic residues. By F. FRANKLIN	- - - -	176
On the equation $x^N - 1 = 0$. By Prof. CAYLEY	- - - -	184
Some inequalities. By W. SEGAR	- - - -	189

GEOMETRY.

On the converses of some theorems in elementary geometry. By C. LEUDESDORF	- - - -	14
The solution of a special case of the problem of the establishment of a correlation between two figures. By J. BRILL	- - - -	57
On an extension of a problem of Pappus's. By ISURUTA KENJI	- - - -	67
Geometrical interpretation of a condition of integrability. By W. BURNSIDE	- - - -	96
The lines of zero length on a surface as curvilinear coordinates. By W. BURNSIDE	- - - -	99
Note on the intersection of lines with curves of an odd degree. By H. G. DAWSON	- - - -	110

On the focals of a quadric surface. By Prof. CAYLEY	113
The solution of a special case of the problem of the establishment of a correlation between two plane figures (second paper). By J. BRILL	151
Note on reciprocal lines. By Prof. CAYLEY	174
Note de géométrie, à propos d'un théorème de M. Stewart. By Prof. MANNHEIM	178
Quaternion proofs of theorems relating to asymptotic lines. By T. MOTODA	188

DIFFERENTIAL AND INTEGRAL CALCULUS, AND FINITE DIFFERENCES.

Note on the differential equation of a conic. By E. B. ELLIOTT	5
On differential expressions which persist in form after the transformation $x = \frac{1}{x'}$, $y = \frac{y'}{x'}$. By E. B. ELLIOTT	7
On Stirling's formula and other interpolation formulas. By G. H. STUART	19

THEORY OF ELLIPTIC FUNCTIONS.

Note on the sums of two series. By Prof. CAYLEY	29
Expansions of K , I , G , E in powers of $k^2 - k'^2$. By J. W. L. GLAISHER	146
On a direct relation between the definite elliptic integrals of the first and second order. By G. F. CHILDS	155
On the expansion of $\frac{G}{K}$, $\frac{K}{G}$, $\frac{I}{E}$, &c., in ascending powers of k^2 . By J. W. L. GLAISHER	164

APPLIED MATHEMATICS.

Note on energy in an elastic solid. By KARL PEARSON	31
The parabolic trajectory. By Prof. GREENHILL	47
The small deformation of curves and surfaces with application to the vibrations of a helix and a circular ring. By J. H. MICHELL	68
On the exhaustion of Neumann's mode of solution for the motion of solids of revolution in liquids, and similar problems. By J. H. MICHELL	83
Vibrations of a string stretched on a surface. By J. H. MICHELL	87
Propagation of energy in the electro-magnetic field. By W. BURNSIDE	98
On the resultant of two finite displacements of a rigid body. By W. BURNSIDE	102
Note on Clapeyron's theorem of the three moments. By KARL PEARSON	129
On the stability of a bent and twisted wire. By J. H. MICHELL	181

MESSENGER OF MATHEMATICS.

A NEW PROOF THAT A GENERAL QUADRIC MAY BE REDUCED TO ITS CANONICAL FORM (THAT IS, A LINEAR FUNCTION OF SQUARES) BY MEANS OF A REAL ORTHOGONAL SUBSTITUTION.

By Prof. J. J. Sylvester.

ALL the proofs that I am acquainted with (and their name is legion) of the possibility of depriving a quadric, in three or more variables, of its mixed terms by a real orthogonal transformation are made to depend on the theorem that the "latent roots" of any symmetrical matrix are all real.

By the latent roots is understood the roots of the determinant expressed by tacking on a variable $-\lambda$ to each term in the diagonal of symmetry to such matrix.

I shall show that the same conclusion may be established *à priori* by purely algebraical ratiocination and without constructing any equation, by the method of cumulative variation. The proof I employ is inductive: *i.e.* if the theorem is true for two or any number of variables I prove that it will be true for one more.

To illustrate the method let us begin with two variables. Consider the form $ax^2 + 2hxy + by^2$.

If in any such form $b = a$, then by an obvious orthogonal transformation, viz. writing $\frac{x+y}{\sqrt{2}}$ and $\frac{x-y}{\sqrt{2}}$ for x and y , the form becomes

$$a(x^2 + y^2) + h(x^2 - y^2),$$

or $(a+h)x^2 + (a-h)y^2.$

Now in general on imposing on x, y any orthogonal infinitesimal substitution, so that

$$x \text{ becomes } x + \varepsilon y,$$

$$y \quad \text{,,} \quad y - \varepsilon x$$

h in the new form becomes $h + (a - b)\varepsilon$, or say $\delta h = (a - b)\varepsilon$, and $\frac{1}{2}\delta(h^2) = (a - b)h\varepsilon$; the variations of a and b need not be set forth.

Let an infinite succession of such transformations be instituted; then either a and b become equal and the orthogonal substitution above referred to reduces the quadric to its canonical form [in which case this one combined with the preceding infinite series of such substitutions may be compounded into a single substitution], or else by giving ε the sign of $(b - a)$ the variation of h^2 may at each step be made negative so that h^2 continually decreases, unless h vanishes. If h does not vanish it must have a minimum value, and this minimum value may be diminished, which involves a contradiction; hence, in the infinite series of substitutions supposed, either a and b become equal or h vanishes, and in either case the quadric is reduced or reducible to its canonical form.

Let us now take the case of three variables x, y, z .

Obviously, by the preceding case, we may make the term involving xy disappear and commence with the initial form

$$ax^2 + by^2 + 2fzx + 2gyz + cz^2.$$

If f or g become zero the quadric may be canonified by virtue of the preceding case.

Again, if $b = a$ by imposing on x, y the orthogonal substitution

$$\frac{g}{\sqrt{f^2 + g^2}}x + \frac{f}{\sqrt{f^2 + g^2}}y \\ - \frac{f}{\sqrt{f^2 + g^2}}x + \frac{g}{\sqrt{f^2 + g^2}}y,$$

the term involving xz will disappear and the final result is the same as if f were zero.

Let us now introduce the infinitesimal orthogonal substitution which changes

$$x \text{ into } x + \varepsilon y + \eta z,$$

$$y \quad \text{,,} \quad -\varepsilon x + y + \theta z,$$

$$z \quad \text{,,} \quad -\eta x - \theta y + z,$$

where $\varepsilon, \eta, \theta$ are supposed to be of the same order of magnitude so that only first powers of them have to be considered.

Then
$$\begin{aligned}\delta f &= (a - c) \eta - g\varepsilon, \\ \delta g &= (b - c) \theta + f\varepsilon,\end{aligned}$$

also the coefficient of $2xy$ becomes $(a - b) \varepsilon - f\theta - g\eta$.

Now whatever η, θ may be, we may determine ε in terms of η, θ so that this may be made to vanish, and the initial form of the quadric will be maintained, provided that b is not equal to a .

Hence instituting an infinite series of these infinitesimal substitutions, provided we do not reach a stage where a and b become equal, we may maintain the original form keeping η, θ arbitrary, and shall have

$$\frac{1}{2} \delta (f^2 + g^2) = (a - c) f \eta + (b - c) g \theta.$$

Suppose a and b to be unequal; therefore $(a - c), (b - c)$ do not vanish simultaneously, and consequently we may make $\delta (f^2 + g^2)$ negative unless at least one of the two quantities f, g vanishes.

If neither of them vanishes $f^2 + g^2$ may be made continually to decrease and will have a minimum other than zero, which involves a contradiction.

Hence the infinite series of infinitesimal orthogonal substitutions may be so conducted that either $a - b$ or one at least of the letters f, g shall become zero; and then two additional orthogonal substitutions at most will serve to reduce the Quadric immediately to its canonical form.

I shall go one step further to the case of four variables x, y, z, t , and then the course of the induction will become manifest. We may, by virtue of what has been shown, take as our quadric

$$ax^2 + by^2 + cz^2 + 2fxt + 2gyt + 2hzt + dt^2.$$

Here, if any one of the mixed terms disappears, the quadric is immediately reducible by the preceding case, and if any two of the grouped pure coefficients a, b, c become equal (as for instance a, b), then by an orthogonal transformation one of the mixed terms (f or g in the case supposed) may be got rid of; so that this supposition merges in the preceding one.

Impose on x, y, z, t an infinitesimal orthogonal substitution, writing

$$\begin{aligned}x + \varepsilon y + \theta z + \lambda t &\text{ for } x, \\ - \varepsilon x + y + \eta z + \mu t &\text{ ,, } y, \\ - \theta x - \eta y + z + \nu t &\text{ ,, } z, \\ - \lambda x - \mu y - \nu z + t &\text{ ,, } t.\end{aligned}$$