THESIS RIEMANN'S P-FUNCTION. DISSERTATION, 1890

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THESIS

RIEMANN'S *P*-FUNCTION

BY

CHARLES H. CHAPMAN

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RIEMANN'S *P*-FUNCTION.

INTRODUCTION.

The following paper was begun at the suggestion of Dr. Craig and has had the benefit of his advice and criticism. It is an attempt to set forth in a clear and readable form the properties of the *P*-function invented by Riemann and treated of in his collected works.

Part I of the present paper is devoted to a restatement in systematic order of the properties of the function as given by Riemann, with full demonstrations of certain points at which he has only hinted, and original proofs of some propositions which he considered almost self-evident, but which admit, and perhaps require, elaborate demonstrations.

Section 4 of Part I is an instance of this kind; the work therein and the table of equivalent functions are my own. The same may be said of Section 5, where, by using an expansion in powers of the variable which is known to be convergent in the proper regions, the ideas are fixed and a certain ease of statement is attained.

In proving that the *P*-function satisfies a linear differential equation of the second order I have followed Riemann, except that certain points in the Theory of Functions have been touched more lightly, as being better known to readers at the present day.

Part II is entirely my own. It is a study of the differential equation satisfied by the *P*-function from the point of view of the modern theory as originated by Fuchs and developed by his illustrious collaborators. The theorems concerning the exponents are stated and proved as properties of the indicial equation in Section 4. Section 3 is an elaborate study of the transformation $x' = \frac{ex + f}{gx + h}$; Section 5 is devoted to obtaining the coefficients of the differential equation; and in subsequent sections the Spherical Harmonics, Toroidal functions and Bessel's functions are expressed as *P*-functions, while, in conclusion, the *P*-function itself is expressed as a Hypergeometric Series.

PART I.

SECTION 1.—*Definition of the P-function.*

Conceive that a function exists which has the following properties:

1. It is finite and continuous throughout the plane of imaginary quantities, except at the points $x = a$, $x = b$, $x = c$.

2. Between any three branches of the function, P', P'', P''', there exists a linear relation with constant coefficients,

$$c'P' + c''P'' + c'''P''' = 0.$$

3. The function may be put in any of the forms

$$C_\alpha P^{(\alpha)} + C_{\alpha'} P^{(\alpha')},\quad C_\beta P^{(\beta)} + C_{\beta'} P^{(\beta')},\quad C_\gamma P^{(\gamma)} + C_{\gamma'} P^{(\gamma')},$$

where C_α, $C_{\alpha'}$, C_β, $C_{\beta'}$, C_γ, $C_{\gamma'}$ are constants, and the expressions $P^{(\alpha)}(x-a)^{-\alpha}$, $P^{(\alpha')}(x-a)^{-\alpha'}$ become neither zero nor infinite when $x=a$; likewise $P^{(\beta)}(x-b)^{-\beta}$, $P^{(\beta')}(x-b)^{-\beta'}$ are neither zero nor infinite for $x=b$, and $P^{(\gamma)}(x-c)^{-\gamma}$, $P^{(\gamma')}(x-c)^{-\gamma'}$ are neither zero nor infinite for $x=c$.

By these properties the P-function is completely defined, except that it contains two arbitrary constants. It is designated by the symbol

$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

SECTION 2.—*The quantities* $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$.

These quantities may be anything whatever, subject to the conditions—

1. None of the differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ shall be an integer.

2. The sum of all the quantities is constantly unity, i. e.,

$$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 1.$$

SECTION 3.—*Properties of the P-function.*

1. The first three vertical columns may be interchanged at pleasure. For, when the defining conditions are applied to the three functions so obtained, no distinction can be observed between them; hence they are identical, provided the conditions actually define a function.

2. In the same way we see that α may be interchanged with α', β with β', and γ with γ'.

3. Let x be replaced by x', a rational linear function of x, so taken that when

$$x=a,\ x'=a',$$
$$x=b,\ x'=b',$$
$$x=c,\ x'=c',$$

then the two functions $P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ and $P\begin{Bmatrix} a', & b', & c' & \\ \alpha, & \beta, & \gamma & x' \\ \alpha', & \beta', & \gamma' & \end{Bmatrix}$ are equal. By this transformation, to be fully developed later on, every P-function may be expressed in terms of another, whose singular points are $0,\ \infty,\ 1$. But every function having the same $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$ may thus be put in the form $P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$, and our definition will then make no distinction between them; that is, all P-functions having the same exponents, $\alpha, \alpha'; \beta, \beta'; \gamma, \gamma'$, may be reduced to the same function $P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$, which may be briefly written $P\begin{Bmatrix} \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$.

According to the linear expression in x which we choose for the variable, the points $0,\ \infty,\ 1$ may appear in six different ways, corresponding to six modes of propagation of the function in the plane of x. They are

$$P\begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} \infty & 0 & 1 & \\ \alpha & \beta & \gamma & \frac{1}{x} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} 1 & \infty & 0 & \\ \alpha & \beta & \gamma & 1-x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};$$

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ \alpha & \beta & \gamma & \frac{x}{x-1} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} \infty & 1 & 0 & \\ \alpha & \beta & \gamma & \frac{x-1}{x} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix};\quad P\begin{Bmatrix} 1 & 0 & \infty & \\ \alpha & \beta & \gamma & \frac{1}{1-x} \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

SECTION 4.—*Transformation of Exponents.*

We shall have, by definition, the product $P^{(\alpha+\delta)}(x-a)^{-\alpha-\delta}$ neither 0 nor ∞ for $x=a$: hence, consistently with all that precedes, we may write, denoting by P_1 a new P-function,

$$P_1^{(\alpha+\delta)} = (x-a)^\delta P^{(\alpha)} (x-b)^{-\delta}, \text{ if we choose; and}$$
$$P_1^{(\alpha'+\delta)} = (x-a)^\delta P^{(\alpha')} (x-b)^{-\delta};\ P_1^{(\beta-\delta)} = (x-b)^{-\delta} P^{(\beta)} (x-a)^\delta;$$
$$P_1^{(\beta'-\delta)} = (x-b)^{-\delta} P^{(\beta')} (x-a)^\delta;\ P_1^{(\gamma)} = (x-a)^\delta (x-b)^{-\delta} P^{(\gamma)};$$
$$P_1^{(\gamma')} = (x-a)^\delta (x-b)^{-\delta} P^{(\gamma')}.$$

Observing that the left-hand members of these equations are the constituent branches of the function $P_1 \begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & x \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{Bmatrix}$, we have the relation

$$P_1 \begin{Bmatrix} a & b & c & \\ \alpha+\delta & \beta-\delta & \gamma & x \\ \alpha'+\delta & \beta'-\delta & \gamma' & \end{Bmatrix} = \left(\frac{x-a}{x-b}\right)^\delta P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

If the P-function be in the reduced form

$$P \begin{Bmatrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & x' \\ \alpha' & \beta' & \gamma' & \end{Bmatrix},$$

then in the region of the point ∞ either branch has the form

$$\left(\frac{1}{x'}\right)^\beta \left(a_0 + \frac{a_1}{x'} + \frac{a_2}{x'^2} + \dots\right) = \left(\frac{1}{x'}\right)^\beta Y^\beta;$$

because $P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$ had in the region of the point b the form

$$(x-b)^\beta [a_0' + a_1'(x-b) + a_2'(x-b)^2 + \dots],$$

and a transformation of such a nature that when $x = b$, x' shall become infinite, can only be of the form $x - b = \frac{1}{x'}$; hence the transformed function in the region of the point ∞ has the form

$$\left(\frac{1}{x}\right)^\beta \left[a_0 + \frac{a_1}{x} + \frac{a_2}{x_2} + \dots\right] = \left(\frac{1}{x}\right)^\beta Y^{(\beta)}.$$

It follows that
$$P \begin{Bmatrix} \alpha & \beta & \gamma & x \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} (1-x)^\epsilon x^\delta$$

will, in the region of the point ∞, be of the form

$$c_\beta \frac{(1-x)^\epsilon x^\delta}{x^\beta} Y^{(\beta)} + c_{\beta'} \frac{(1-x)^\epsilon x^\delta}{x^{\beta'}} Y^{(\beta')},$$

where $Y^{(\beta)}$ and $Y^{(\beta')}$ are neither zero nor infinite for $x=\infty$. Clearly then, putting this in the form $c_\beta P^{(\beta)}+c_{\beta'}P^{(\beta')}$, we see that $P^{(\beta)}x^{\beta-\delta-\epsilon}$ and $P^{(\beta')}x^{\beta'-\delta-\epsilon}$ are neither 0 nor ∞ at the point ∞; hence

$$P\begin{Bmatrix}\alpha & \beta & \gamma & x\\ \alpha' & \beta' & \gamma' & \end{Bmatrix}(1-x)^{\epsilon}x^{\delta}=P\begin{Bmatrix}\alpha+\delta & \beta-\delta-\varepsilon & \gamma+\varepsilon & x\\ \alpha'+\delta & \beta'-\delta-\varepsilon & \gamma'+\varepsilon & \end{Bmatrix};$$

the first and last exponents being transformed by the rule found above.

Here δ and ε may have any values whatever, and this remark permits us to draw the following inference:

The values of any two of the exponents may be changed at pleasure by introducing proper multipliers; but the sum $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'$ must remain unchanged, and always equal to 1. The differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ must also remain unaltered in absolute magnitude. In other words, the product of a P-function by factors which fulfil the above conditions may be expressed as a P-function.

Again, P-functions, in which the differences $\alpha-\alpha'$, $\beta-\beta'$, $\gamma-\gamma'$ are the same, can differ only by determinate factors, as the following table will more fully illustrate. The transformations involved will be considered in Part II.

$$P\begin{pmatrix}\alpha & \beta & \gamma & x\\ \alpha' & \beta' & \gamma' & \end{pmatrix} = \begin{cases}(1-x)^{\gamma}x^{\alpha}P\begin{pmatrix}0 & \beta+\alpha+\gamma & 0 & x\\ \alpha'-\alpha & \beta'+\alpha+\gamma & \gamma'-\gamma & \end{pmatrix},\\ (1-x)^{\gamma'}x^{\alpha'}P\begin{pmatrix}\alpha-\alpha' & \beta+\alpha'+\gamma' & \gamma-\gamma' & x\\ 0 & \beta'+\alpha'+\gamma' & 0 & \end{pmatrix},\\ (1-x)^{\gamma}x^{\alpha'}P\begin{pmatrix}\alpha-\alpha' & \beta+\alpha'+\gamma & 0 & x\\ 0 & \beta'+\alpha'+\gamma & \gamma'-\gamma & \end{pmatrix},\\ (1-x)^{\gamma'}x^{\alpha}P\begin{pmatrix}0 & \beta+\alpha+\gamma' & \gamma-\gamma' & x\\ \alpha'-\alpha & \beta'+\alpha+\gamma' & 0 & \end{pmatrix}.\end{cases}$$

$$P\begin{pmatrix}\beta & \alpha & \gamma & \frac{1}{x}\\ \beta' & \alpha' & \gamma' & \end{pmatrix} = \begin{cases}\left(\frac{x-1}{x}\right)^{\gamma}\frac{1}{x^{\beta}}\;P\begin{pmatrix}0 & \alpha+\beta+\gamma & 0 & \frac{1}{x}\\ \beta'-\beta & \alpha'+\beta+\gamma & \gamma'-\gamma & \end{pmatrix},\\ \left(1-\frac{1}{x}\right)^{\gamma'}\frac{1}{x^{\beta'}}P\begin{pmatrix}\beta-\beta' & \alpha+\beta'+\gamma' & \gamma-\gamma' & \frac{1}{x}\\ 0 & \alpha'+\beta'+\gamma' & 0 & \end{pmatrix},\\ \left(1-\frac{1}{x}\right)^{\gamma}\frac{1}{x^{\beta'}}P\begin{pmatrix}\beta-\beta' & \alpha+\beta'+\gamma & 0 & \frac{1}{x}\\ 0 & \alpha'+\beta'+\gamma & \gamma'-\gamma & \end{pmatrix},\\ \left(1-\frac{1}{x}\right)^{\gamma'}\frac{1}{x^{\beta}}P\begin{pmatrix}0 & \alpha+\beta+\gamma' & \gamma-\gamma' & \frac{1}{x}\\ \beta'-\beta & \alpha'+\beta+\gamma' & 0 & \end{pmatrix},\end{cases}$$