INVARIANTS AND EQUATIONS ASSOCIATED WITH THE GENERAL LINEAR DIFFERENTIAL EQUATION

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INVARIANTS AND EQUATIONS

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General Linear Differential Equation

THESIS PRESENTED FOR THE DEGREE OF PH. D.

BY

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INTRODUCTION.

The formation of functions, associated with differential equations and analogous to the invariants of algebraic quantics, has occupied the attention of several mathematicians for some years, because of their great value in leading to practical as well as theoretical solutions of such equations.

Starting with the work of M. Laguerre and of Professor Brioschi, M. Halphen, in two important memoirs,* indicated a method for the formation of invariants, but involving very difficult analysis. He derived the two simplest invariants for the cubic and quartic and such derivatives as may be deduced from them. For this purpose he, by means of the transformation $Y = ye^{-\int \frac{R_1}{R_0} dx}$, brings the equation to a form having zero for the coefficient of the second term.

Meanwhile Mr. Forsyth, starting with the letter of Professor Brioschi, prepared a very valuable memoir,† in which, by means of the following transformations, he obtains a canonical form in which the coefficients of both the second and third terms vanish. This may be stated as follows:

When the linear differential equation

$$\frac{d^n y}{dx^n} + \binom{n}{2} P_2 \frac{d^{n-2}y}{dx^{n-2}} + \binom{n}{3} P_3 \frac{d^{n-3}y}{dx^{n-3}} + \binom{n}{4} P_4 \frac{d^{n-4}y}{dx^{n-4}} + \dots + P_n = 0$$

* "Mémoire sur la réduction des équations differentielles linéaires aux formes integrables" (*Mémoires des Savants Étrangers*, Vol. 28, No. 1, 301 pp., 1880). Also, "Sur les invarients des équations differentielles linéaires du quatrième ordre" (*Acta Math.*, Vol. 3, 1883, pp. 325–380).

† "Invariants, Covariants and Quotient Derivatives associated with Linear Differential Equations."—*Philosophical Transactions of the Royal Society of London*, Vol. 179 (1888), A, pp. 377–489.

has its dependent variable y transformed to u by the equation $y = u\lambda$, λ being a function of x and its independent variable changed from x to z, where z and λ are determined by

$$\lambda = \varphi^{n-1}, \quad \frac{dz}{dx} = \varphi^{-2}, \tag{1}$$

$$\frac{d^2\varphi}{dx^2} + \frac{3}{n+1} P_2\varphi = 0, \tag{2}$$

the transformed in u is in the canonical form

$$\frac{d^n u}{dz^n} + \binom{n}{3} Q_3 \frac{d^{n-3}u}{dz^{n-3}} + \binom{n}{4} Q_4 \frac{d^{n-4}u}{dz^{n-4}} + \ldots + Q_n = 0,$$

$\binom{n}{r}$ being the binomial coefficient $\dfrac{n!}{r!\,n-r!}$.

The coefficients P and Q of these equations are so connected that there exist $n - 2$ algebraically independent functions $\theta_\sigma(x)$ of the coefficients P and their derivatives which are such that, when the same function $\theta_\sigma(z)$ is formed of the coefficients Q and their derivatives, the equation

$$\theta_\sigma(x) = (z')^\sigma \theta_\sigma(z) \tag{3}$$

is identically satisfied. For this form of the differential equation

$$\theta_\sigma(z) \equiv Q_\sigma + \frac{\sigma}{2} \sum_{r=1}^{r=\sigma-3} (-1)^r a_{r,\sigma} \frac{d^r Q_{\sigma-r}}{dz^r},$$

where

$$a_{r,\sigma} = \frac{\sigma-1!\,\sigma-2!\,2\sigma-r-2!}{r!\,2\sigma-3!\,\sigma-r!\,\sigma-r-1!}.$$

Thus $\theta_\sigma(z)$ is independent of the order of the equation. In this z is completely determined by equations (1) and (2). But there may be difficulties in the way of solving (2), and thus it is desirable to form the invariants for the uncanonical form of the equation.

For this purpose Mr. Forsyth establishes relations between the coefficients P and Q for the case in which z, being arbitrary, is given the value $x + \varepsilon\mu$, where ε is so small that the square and higher powers may be neglected, and μ is an arbitrary non-constant function of x. These relations are expressed thus:

$$\left.\begin{aligned} Q_s = P_s(1 - s\epsilon\mu') - \frac{\epsilon}{2}\sum_{\theta=0}^{\theta=s-1}\Bigg[\frac{s!}{\theta!\,\overline{s-\theta+1}!} \\ \{n(s-\theta-1)+s+\theta-1\}\,P_\theta\,\frac{d^{s-\theta+1}\mu}{dx^{s-\theta+1}}\Bigg]\end{aligned}\right\}. \quad (5)$$

These relations are fully developed in Mr. Forsyth's memoir; also in Dr. Craig's excellent work* they will be found, and such a general treatment of the whole subject of differential equations and differential quantics as makes the work an invaluable help and guide to any student of the subject.

Then we derive

$$\left.\begin{aligned} \frac{d^rQ_s}{dx^r} = \frac{d^rP_s}{dx^r}\{1-(r+s)\epsilon\mu'\} - s\epsilon P_s\frac{d^{r+1}\mu}{dx^{r+1}} \\ -\epsilon\sum_{m=1}^{m=r-1}\Bigg[\frac{r!}{m!\,\overline{r-m+1}!}\{s(r+1) \\ -m(s-1)\}\frac{d^mP}{dx^m}\,\frac{d^{r-m+1}\mu}{dx^{r-m+1}}\Bigg] \\ -\frac{\epsilon}{2}\sum_{\theta=0}^{\theta=s-1}\Bigg[\frac{s!}{\theta!\,\overline{s-\theta+1}!}\{n(s-\theta-1) \\ +s+\theta-1\}\frac{d^r}{dx^r}\left(P_\theta\frac{d^{s-\theta+1}\mu}{dx^{s-\theta+1}}\right)\Bigg]\end{aligned}\right\}. \quad (6)$$

The only invariants that have been formed, so far as I know, are θ_3, θ_4, θ_5, θ_6 and θ_7, where θ_r is the invariant of the rth order of an equation of order n.

In Section I of this thesis the general invariant θ_s is considered, and it is there shown that in the non-linear part every term is of the form ABC. Where A is a number, B is a function of P_2 and its derivatives, and C is an invariant or the derivative of an invariant with suffix differing from s by an even number. When s is even C may be a number.

Section II deals with the coefficients of θ_s, giving some

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* Treatise on Linear Differential Equations. By Thomas Craig, Ph.D. Vol. I.

general expressions by which they may be calculated for any value of *s*.

Section III treats of associate variables and associate equations, showing which are identical and which may not be.

Dr. Craig having discovered that the condition for the self-adjointness of the sextic and octic was that their invariants with odd suffix all vanish, suggested to me the general theorem announced in his treatise, pp. 293–295. The proof given at that time only applied to equations in Mr. Forsyth's canonical form. By aid of what is established in Section I, it is shown to apply also to equations in any form.

A fuller history of the subject will be found in the works to which reference has been made.

This paper was not only suggested by Dr. Craig, but has had his valuable criticism.

Section I.

THE FORM OF THE GENERAL LINEAR PRIME INVARIANT θ_s.

Since θ_s has only a linear part when P_2 vanishes, its form must be as follows:

$$
\begin{array}{llr}
 & [AP_s + BP'_{s-1} + CP''_{s-2} + \ldots + WP_2^{(s-2)} &] \\
+ & [P_2\{a_2\theta_{s-2} \quad + a_3\theta'_{s-3} + a_4\theta''_{s-4} + \ldots + \quad a_{s-2}P_2^{(s-4)} & \}] \\
+ & [P'_2\{ \quad b_3\theta_{s-3} + b_4\theta'_{s-4} + \ldots + \quad b_{s-3}P_2^{(s-6)} & \}] \\
+ & [P''_2\{ \quad c_4\theta_{s-4} + c_5\theta'_{s-5} + \ldots + c_{s-2}P_2^{(s-6)} & \}] \\
+ & [P_2^2\{ \quad d_4\theta_{s-4} + d_5\theta'_{s-5} + \ldots + d_{s-2}P_{s-2}^{(s-6)} & \}] \\
+ & [P'''_2\{ \ \ldots\ldots\ldots\ldots\ldots\ldots &] \\
+ & [\text{etc.} \ \ldots\ldots\ldots\ldots\ldots\ldots &] \\
+ & [\text{etc.} \ \ldots\ldots\ldots\ldots\ldots\ldots &] \\
 & \text{etc., etc.} \ \ldots\ldots\ldots\ldots\ldots\ldots &
\end{array}
$$

In this (r) is the differential index, so that

$$P_2^{(r)} \equiv \frac{d^r P_2}{dx^r}, \qquad \theta^{(r)}_{s-x} \equiv \frac{d^r \theta_{s-x}}{dx^r} \ldots$$

The sum of the suffixes and differential indices, it will be noticed, equals s for every term; that is, θ_s possesses a kind of homogeneity.* s is called the index or dimension number of θ_s; the dimension number of ${P_2^{(p)}}^r\theta^{\mu}_{s-x}$ being

$$(p + 2)r + \mu + s - x.$$

Denoting the terms within the square parenthesis by L, α, β, γ, δ, etc., then $\theta_s \equiv L + \alpha + \beta + \gamma + \delta + \ldots$

The notation used here will be nearly that used by Mr. Forsyth, but to simplify the work the μ's and their derivatives arising from $z = x + \epsilon\mu$ will be dropped, that is, they will be

* Philosophical Transactions, Vol. 179 (1888) A, pp. 391–92.

treated as unity, when the result will not be changed by doing so. Also $P_r^{(0)} \equiv \frac{d^0 P_r}{dx^0}$ will be considered $\equiv$ with P_r.

The general form of the terms in L is

$$(-1)^\tau \frac{s!\,\overline{s-2}!\,\overline{2s-\tau-2}!}{2.\tau!\,\overline{s-\tau}!\,\overline{s-\tau-1}!\,\overline{2s-3}!}\, P_{s-\tau}^{(\tau)},\ \tau = 0, 1, 2 \dots s-2. \quad (a)$$

I shall now show that when κ is odd each of the numerical coefficients a_κ, b_κ, c_κ, d_κ, etc., of the non-linear part of θ_s equals zero.

From page 4 of the introduction we have

$$\theta_s(x) = z'^s\, \theta_s(z) = (1 - s\epsilon\mu')\, \theta_s(z)$$

identically satisfied. If in the right member of this identity the Q's and their derivatives are replaced by their values in terms of the P's and their derivatives, as expressed by formulae (5) and (6) (page 5, introduction), then the terms of dimension 's' in each member cancel, those of dimension '$s-1$' furnish the numerical coefficients in the linear part L, and there remain terms of dimension equal to and less than $s-2$ with which we may determine the coefficients of the non-linear part.

Remembering the convention $P_r^{(0)} \equiv P_r$, formulae (5) and (6) are included in

$$\left.\begin{aligned} \frac{d^r Q_s}{dz^r} &= P_s^{(r)}\{1 - (r+s)\epsilon\mu'\} \\ &\quad - \epsilon \sum_{m=0}^{m=r-1} \left[\frac{r!}{m!\,\overline{r-m+1}!}\{s(r-m+1) \right. \\ &\qquad\qquad \left. + m\}\, P_s^{(m)} \mu^{(r-m+1)}\right] \\ &\quad - \frac{\epsilon}{2} \sum_{\theta=0}^{\theta=s-1} \left[\frac{s!}{\theta!\,\overline{s-\theta+1}!}\{n(s-\theta-1) \right. \\ &\qquad\qquad \left. + s + \theta - 1\}(P_\theta \mu^{(s-\theta+1)})^{(r)}\right] \\ &\qquad\qquad r = 0, 1, 2, 3 \dots s \end{aligned}\right\} \quad . \quad (7)$$

Also, differentiating the invariants, we find