SOLUTIONS OF THE EXAMPLES APPENDED TO A TREATISE ON THE MOTION OF A RIGID BODY

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EXAMPLES

APPENDED TO A TREATISE

ON

THE MOTION OF A RIGID BODY.

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EXAMPLES

OF THE

MOTION OF A RIGID BODY.

The references in the following solutions are to articles in the treatise which they are intended to follow.

SECTION I.

GEOMETRICAL PROPERTIES OF A RIGID BODY.

1. (α) If x', y' be the co-ordinates of m referred to axes originating in the centre and parallel to those to which x, y refer,

$$x = a + x', \quad y = b + y'.$$

$$\therefore \Sigma(mxy) = \Sigma(m) . ab + a\Sigma(my') + b\Sigma(mx') + \Sigma(mx'y').$$

Now

$$\left.\begin{array}{l}\Sigma(mx') = 0\\ \Sigma(my') = 0\end{array}\right\},$$

by properties of the centre of gravity.

Also $\Sigma(mx'y') = 0$, since for any given value of y' the values of x' form pairs equal in magnitude and opposite in sign.

$$\therefore \Sigma(mxy) = \Sigma(m) . ab = Mab.$$

Since $z = 0$ for every point of the mass,

$$\therefore \Sigma(myz) = 0, \quad \Sigma(mxz) = 0.$$

The same method applies to (β), (ϵ), (ζ).

(γ) Since $z = 0$ for every point of the mass,

$$\Sigma(myz) = 0, \quad \Sigma(mxz) = 0.$$

Let $OB = a$, $OA = b$. (fig. 1).

If P be a point of the lamina whose co-ordinates are x, y, the area of an element at P, whose sides are parallel to the axes of co-ordinates, is $dx.dy$, and this may also represent the mass of a corresponding element of the lamina if the mass of an unit of area of the lamina be unity;

$$\therefore \Sigma(mxy) = \int_x\int_y xy.$$

Now first the summation of the values of this function for elements which lie along the line MQ and form an elementary strip of the body in that direction, is equivalent to integrating with respect to y from $y = 0$ to $y = MQ = (a - x) \tan B$.

Hence for such a strip $\Sigma(myx)$ becomes

$$\tfrac{1}{2}\int_x x(a - x)^2 \tan^2 B$$

$$= \tfrac{1}{2}\int_x (a^2 x - 2ax^2 + x^3) \tan^2 B.$$

Secondly, the required summation will be completed by adding the values of the function for such strips as those just considered, as they range from OA to B, and this amounts to integrating the expression just obtained from $x = 0$ to $x = a$;

$$\therefore \text{ finally, } \Sigma(mxy) = \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4}) a^4 \tan^2 B$$

$$= \tfrac{1}{24} a^2 b^2;$$

while M, the mass, has by virtue of the units adopted been represented by $\frac{1}{2}ab$;

$$\therefore \Sigma(mxy) = \tfrac{1}{12} Mab.$$

(δ) Let l, m, n be the cosines of the inclinations of this line to the co-ordinate axes; r the distance of a point x, y, z in the line from its centre, and let the mass of an element at this point be represented by its length dr.

The co-ordinates of the centre of the line being

$$\frac{a + a'}{2}, \quad \frac{b + b'}{2}, \quad \frac{c + c'}{2},$$

we have

$$\Sigma(myz) = \Sigma\left(\frac{b+b'}{2} + rm\right)\left(\frac{c+c'}{2} + rn\right) dr$$

$$= M \cdot \frac{b+b'}{2} \cdot \frac{c+c'}{2} + mn \int_r^{-R} {}^{R} r^2,$$

if $2R$ be the whole length of the line,

$$= M \cdot \frac{b+b'}{2} \cdot \frac{c+c'}{2} + \tfrac{2}{3} mn \cdot R^3,$$

$$= M\left\{\frac{(b+b')(c+c')}{4} + \frac{(b-b')(c-c')}{12}\right\},$$

$$= \tfrac{1}{12} M\left\{4(bc+b'c') + 2(bc'+b'c)\right\},$$

$$= \tfrac{1}{6} M\left\{2(bc+b'c') + bc' + b'c\right\}.$$

So $\Sigma(mxz) = \tfrac{1}{6} M\left\{2(ac+a'c') + ac' + a'c\right\}$.

$\Sigma(mxy) = \tfrac{1}{6} M\left\{2(ab+a'b') + ab' + a'b\right\}$.

(η) This is an instance where the evaluation of a function is expedited by transforming it to other co-ordinate axes.

Let x', y' be the co-ordinates of the point x, y, when referred to other rectangular axes in the plane of xy so that x' is in the axis of the cone;

$$\therefore \left.\begin{aligned} x &= x' \cos\alpha - y' \sin\alpha \\ y &= x' \sin\alpha + y' \cos\alpha \end{aligned}\right\}.$$

The facility of the limits of integration in x' and y' recommends them in preference to x and y. Changing the integration to the former, we have

$$\Sigma(mxy) = \int_x\int_y xy$$

$$= \int_{x'}\int_{y'} xy\left(\frac{dx}{dx'} \cdot \frac{dy}{dy'} - \frac{dx}{dy'} \cdot \frac{dy}{dx'}\right)$$

(Gregory's *Ex.* Chap. III., or Moigno, Vol. II. p. 214)

$$= \int_{x}\int_{y} xy$$

$$= \int_{x'}\int_{y'} (x'^2 - y'^2) \,.\, \sin\alpha\cos\alpha$$

$$= \int_{x'}\int_{y'} \{(x'^2 + z'^2) - (y'^2 + z'^2)\} \sin\alpha \,.\, \cos\alpha.$$

Now $\int_{x'}\int_{y'} (x'^2 + z'^2)$ is the moment of inertia of the body about a line through its vertex perpendicular to its axis, and therefore $= \frac{3}{20} M\,(4a^2 + r^2)$, (Walton's *Examples.*)

and $\int_{x'}\int_{y'} (y'^2 + z'^2)$ is the moment of inertia of the cone about its axis and $= \frac{3}{10} Mr^2$,

$$\therefore\ \Sigma(mxy) = M \sin\alpha\cos\alpha\,\{\tfrac{3}{5}a^2 - \tfrac{3}{20}r^2\}.$$

2. If affixes denote quantities belonging to different particles of the body,

$\Sigma(my^2)\Sigma(mz^2)$ when developed consists of

(1) terms expressed by $\Sigma(m_1^2 y_1^2 z_1^2)$

produced by the multiplication of terms in the two factors belonging to the same particle,

(2) terms expressed by $\Sigma m_1 m_2 (y_1^2 z_2^2 + z_1^2 y_2^2)$,

produced by the union of terms belonging to different particles.

Again, $(\Sigma myz)^2$ will similarly be represented by

$$\Sigma(m_1^2 y_1^2 z_1^2) + 2\Sigma(m_1 m_2 y_1 z_1 y_2 z_2),$$

therefore the difference is $\Sigma m_1 m_2 (y_1 z_2 - z_1 y_2)^2$, a quantity necessarily positive.

$$\therefore\ \Sigma(my^2)\,.\,\Sigma(mz^2) > (\Sigma myz)^2.$$

So $$\Sigma(mx^2)\Sigma(mz^2) > (\Sigma mxz)^2,$$
$$\Sigma(mx^2)\Sigma(my^2) > (\Sigma mxy)^2.$$

3. Let the arc AB (fig. 2) be doubled; then the moment of inertia of the whole BAC, whose mass is $2M$, will be double that of AB, which is required.

Let G be the centre of gravity of BAC.

Then $2Mr^2$ being the moment of inertia of BAC about an axis perpendicular to its plane through O the centre,

$$\text{moment required} = \tfrac{1}{2}(2Mr^2 - 2M\,.\,OG^2 + 2M\,.\,AG^2), \quad (8).$$
$$= Mr^2 - Mr\,.\,(OG - AG),$$
$$= 2Mr^2 - 2Mr\,.\,OG,$$
$$= 2Mr^2 - 2Mr^2\frac{\sin\theta}{\theta}.$$

The mode of obtaining the result in this question exemplifies the manner of finding the moment of inertia when that about another parallel axis is known, viz. by passing through a parallel axis at the centre of gravity. The moment of inertia of the axis through O being known by inspection, we get in succession: (1) that about a parallel axis at G, (2) that about a parallel axis at A.

4. The moment of inertia of the arc of the circle about an axis through its centre perpendicular to its plane is Mr^2, every particle having the same distance r from this axis. But this moment of inertia is the sum of those about two perpendicular diameters of the circle (9), and these latter are equal to one another, therefore either of them $= \frac{1}{2}Mr^2$.

5. Let α, β be the semi-axes of one stratum. A homogeneous spheroid bounded by this surface will have about its axis of figure the moment of inertia $\frac{8}{15}\pi\rho\alpha^4\beta$, ρ being its density, or $\frac{8}{15}\pi\rho\alpha^5\sqrt{1-e^2}$.

Now if this expression be differentiated with regard to a, ρ and e remaining unaffected, we obtain the moment of inertia of an infinitely thin homogeneous shell between the surface above mentioned and a contiguous one. The result is

$$\tfrac{8}{3}\pi\rho a^4\sqrt{1-e^2}\,.\,da.$$

The integral of this from zero to a is the moment of inertia of the body formed by an assemblage of such shells, ρ being a function of a in this integration according to the law of density of the solid.

$$\text{Ex.}\quad \int_a^0 \rho a^4 = c\int_a^0 a^3 \sin na$$

$$= -\frac{c}{n}\int_a^0 a^3 \frac{d(\cos na)}{da}$$

$$= -\frac{c}{n}a^3\cos na + \frac{3c}{n}\int_a^0 a^2\cos na.$$

$$\int_a^0 a^2\cos na = \frac{1}{n}\int_a^0 a^2\frac{d(\sin na)}{da}$$

$$= \frac{1}{n}a^2\sin na - \frac{2}{n}\int_a^0 a\sin na\,;$$

$$\int_a^0 a\sin na = -\frac{1}{n}a\cos na + \frac{1}{n^2}\sin na\,;$$

$$\therefore\ \int_a^0 a^2\cos na = \frac{a^2}{n}\sin na + \frac{2a}{n^2}\cos na - \frac{2}{n^3}\sin na,$$

$$\int_a^0 \rho a^4 = -\frac{ca^3}{n}\cos na + \frac{3ca^2}{n^2}\sin na + \frac{6ca}{n^3}\cos na - \frac{6c}{n^4}\sin na.$$

Therefore the moment of inertia required is

$$\tfrac{8}{3}\frac{\pi c\sqrt{1-e^2}}{n^4}\left\{3(a^2n^2-2)\sin na + (6an - a^3n^3)\cos na\right\}.$$