AN ELEMENTARY TREATISE ON FOURIER'S SERIES AND SPHERICAL, CYLINDRICAL, AND ELLIPSOIDAL HARMONICS, WITH APPLICATIONS TO PROBLEMS IN MATHEMATICAL PHYSICS, PP. 1-283

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649053841

An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics, pp. 1-283 by William Elwood Byerly

Edited by Trieste Publishing Pty Ltd.
Cover @ 2017

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WITH

APPLICATIONS TO PROBLEMS IN MATHEMATICAL PHYSICS.

BY

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BOSTON, U.S.A.:
GINN & COMPANY, PUBLISHERS,
1893.

PREFACE.

About ten years ago I gave a course of lectures on Trigonometric Series, following closely the treatment of that subject in Riemann's "Partielle Differentialgleichungen," to accompany a short course on The Potential Function, given by Professor B. O. Peirce.

My course has been gradually modified and extended until it has become an introduction to Spherical Harmonics and Bessel's and Lamé's Functions.

Two years ago my lecture notes were lithographed by my class for their own use and were found so convenient that I have prepared them for publication, hoping that they may prove useful to others as well as to my own students. Meanwhile, Professor Peirce has published his lectures on "The Newtonian Potential Function" (Boston, Ginn & Co.), and the two sets of lectures form a course (Math. 10) given regularly at Harvard, and intended as a partial introduction to modern Mathematical Physics.

Students taking this course are supposed to be familiar with so much of the infinitesimal calculus as is contained in my "Differential Calculus" (Boston, Ginn & Co.) and my "Integral Calculus" (second edition, same publishers), to which I refer in the present book as "Dif. Cal." and "Int. Cal." Here, as in the "Calculus," I speak of a "derivative" rather than a "differential coefficient," and use the notation D_x instead of $\frac{\delta}{\delta x}$ for "partial derivative with respect to x."

The course was at first, as I have said, an exposition of Riemann's "Partielle Differentialgleichungen." In extending it, I drew largely from Ferrer's "Spherical Harmonics" and Heine's "Kugelfunctionen," and was somewhat indebted to Todhunter ("Functions of Laplace, Bessel, and Lamé"), Lord Rayleigh ("Theory of Sound"), and Forsyth ("Differential Equations").

In preparing the notes for publication, I have been greatly aided by the criticisms and suggestions of my colleagues, Professor B. O. Peirce and Dr. Maxime Bôcher, and the latter has kindly contributed the brief historical sketch contained in Chapter IX.

W. E. BYERLY.

Cambridge, Mass., Sept. 1893.

CHAPTER I.

INTRODUCTION.

1. In many important problems in mathematical physics we are obliged to deal with *partial differential equations* of a comparatively simple form.

For example, in the Analytical Theory of Heat we have for the change of temperature of any solid due to the flow of heat within the solid, the equation

$$D_t u = a^2(D_x^2 u + D_y^2 u + D_z^2 u),^* \qquad [\text{I}]$$

where u represents the temperature at any point of the solid and t the time.

In the simplest case, that of a slab of infinite extent with parallel plane faces, where the temperature can be regarded as a function of one coördinate, [I] reduces to

$$D_t u = a^2 D_x^2 u, \qquad [\text{II}]$$

a form of considerable importance in the consideration of the problem of the cooling of the earth's crust.

In the problem of the permanent state of temperatures in a thin rectangular plate, the equation [I] becomes

$$D_x^2 u + D_y^2 u = 0. \qquad [\text{III}]$$

In *polar* or *spherical coördinates* [I] is less simple, it is

$$D_t u = \frac{a^2}{r^2}\left[D_r(r^2 D_r u) + \frac{1}{\sin\theta} D_\theta(\sin\theta\, D_\theta u) + \frac{1}{\sin^2\theta} D_\phi^2 u\right]. \qquad [\text{IV}]$$

In the case where the solid in question is a sphere and the temperature at any point depends merely on the distance of the point from the centre [IV] reduces to

$$D_t(ru) = a^2 D_r^2(ru). \qquad [\text{V}]$$

In *cylindrical coördinates* [I] becomes

$$D_t u = a^2\left[D_r^2 u + \frac{1}{r} D_r u + \frac{1}{r^2} D_\phi^2 u + D_z^2 u\right]. \qquad [\text{VI}]$$

In considering the flow of heat in a cylinder when the temperature at any point depends merely on the distance r of the point from the axis [VI] becomes

$$D_t u = a^2(D_r^2 u + \frac{1}{r} D_r u). \qquad [\text{VII}]$$

* For the sake of brevity we shall often use the symbol ∇^2 for the operation $D_x^2 + D_y^2 + D_z^2$; and with this notation equation [I] would be written $D_t u = a^2 \nabla^2 u$.

In Acoustics in several problems we have the equation

$$D_t^2 y = a^2 D_x^2 y; \qquad [\text{VIII}]$$

for instance, in considering the transverse or the longitudinal vibrations of a stretched elastic string, or the transmission of plane sound waves through the air.

If in considering the transverse vibrations of a stretched string we take account of the resistance of the air [VIII] is replaced by

$$D_t^2 y + 2kD_t y = a^2 D_x^2 y. \qquad [\text{IX}]$$

In dealing with the vibrations of a stretched elastic membrane, we have the equation

$$D_t^2 z = c^2(D_x^2 z + D_y^2 z), \qquad [\text{X}]$$

or in *cylindrical coördinates*

$$D_t^2 z = c^2(D_r^2 z + \frac{1}{r} D_r z + \frac{1}{r^2} D_\phi^2 z). \qquad [\text{XI}]$$

In the theory of *Potential* we constantly meet Laplace's Equation

$$D_x^2 V + D_y^2 V + D_z^2 V = 0 \qquad [\text{XII}]$$

or

$$\nabla^2 V = 0$$

which in *spherical coördinates* becomes

$$\frac{1}{r^2}\left[r D_r^2(rV) + \frac{1}{\sin\theta} D_\theta(\sin\theta D_\theta V) + \frac{1}{\sin^2\theta} D_\phi^2 V \right] = 0, \qquad [\text{XIII}]$$

and in *cylindrical coördinates*

$$D_r^2 V + \frac{1}{r} D_r V + \frac{1}{r^2} D_\phi^2 V + D_z^2 V = 0. \qquad [\text{XIV}]$$

In *curvilinear coördinates* it is

$$h_1 h_2 h_3 \left[D_{\rho_1}\left(\frac{h_1}{h_2 h_3} D_{\rho_1} V\right) + D_{\rho_2}\left(\frac{h_2}{h_3 h_1} D_{\rho_2} V\right) + D_{\rho_3}\left(\frac{h_3}{h_1 h_2} D_{\rho_3} V\right) \right] = 0; \qquad [\text{XV}]$$

where $f_1(x,y,z) = \rho_1, \ f_2(x,y,z) = \rho_2, \ f_3(x,y,z) = \rho_3$

represent a set of surfaces which cut one another at right angles, no matter what values are given to ρ_1, ρ_2, and ρ_3; and where

$$h_1^2 = (D_x\rho_1)^2 + (D_y\rho_1)^2 + (D_z\rho_1)^2$$

$$h_2^2 = (D_x\rho_2)^2 + (D_y\rho_2)^2 + (D_z\rho_2)^2$$

$$h_3^2 = (D_x\rho_3)^2 + (D_y\rho_3)^2 + (D_z\rho_3)^2,$$

and, of course, must be expressed in terms of ρ_1, ρ_2, and ρ_3.

If it happens that $\nabla^2\rho_1 = 0$, $\nabla^2\rho_2 = 0$, and $\nabla^2\rho_3 = 0$, then Laplace's Equation [XV] assumes the very simple form

$$h_1^2 D_{\rho_1}^2 V + h_2^2 D_{\rho_2}^2 V + h_3^2 D_{\rho_3}^2 V = 0. \qquad [\text{XVI}]$$

2. A *differential equation* is an equation containing derivatives or differentials with or without the primitive variables from which they are derived.

The *general solution* of a differential equation is the equation expressing the most general relation between the primitive variables which is consistent with the given differential equation and which does not involve differentials or derivatives. A general solution will always contain *arbitrary* (*i. e.*, undetermined) *constants* or *arbitrary functions.*

A *particular solution* of a differential equation is a relation between the primitive variables which is consistent with the given differential equation, but which is less general than the general solution, although included in it.

Theoretically, every particular solution can be obtained from the general solution by substituting in the general solution particular values for the arbitrary constants or particular functions for the arbitrary functions; but in practice it is often easy to obtain particular solutions directly from the differential equation when it would be difficult or impossible to obtain the general solution.

3. If a problem requiring for its solution the solving of a differential equation is *determinate*, there must always be given in addition to the differential equation enough outside conditions for the determination of all the arbitrary constants or arbitrary functions that enter into the general solution of the equation; and in dealing with such a problem, if the differential equation can be readily solved the natural method of procedure is to obtain its general solution, and then to determine the constants or functions by the aid of the given conditions.

It often happens, however, that the general solution of the differential equation in question cannot be obtained, and then, since the problem *if determinate* will be solved if by any means a solution of the equation can be found which will also satisfy the given outside conditions, it is worth while to try to get *particular solutions* and so to combine them as to form a result which shall satisfy the given conditions without ceasing to satisfy the differential equation.

4. A differential equation is *linear* when it would be of the first degree if the dependent variable and all its derivatives were regarded as algebraic unknown quantities. If it is linear and contains no term which does not involve the dependent variable or one of its derivatives, it is said to be *linear* and *homogeneous.*

All the differential equations collected in Art. 1 are linear and homogeneous.

5. *If a value of the dependent variable has been found which satisfies a given homogeneous, linear, differential equation, the product formed by multiplying this value by any constant will also be a value of the dependent variable which will satisfy the equation.*