ON THE APPLICATION OF A NEW ANALYTIC METHOD TO THE THEORY OF CURVES AND CURVED SURFACES

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BY THE

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AND CHAPLAIN TO THE MOST NOBLE THE MARQUESS OF LANSDOWNE, ETC.

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TO THE REVEREND

FRANC SADLEIR, D.D.,

PROVOST OF TRINITY COLLEGE, DUBLIN,

THE FOLLOWING PAGES ARE DEDICATED

AS A TRIBUTE OF RESPECT,

BY THE

AUTHOR.

ADVERTISEMENT.

I FEAR that brevity and compression have been but too much studied in the following essay, but the necessity of comprising the whole matter in a small compass, and the pressure of other avocations, will plead, I hope, a sufficient apology.

From the same cause I have been obliged to omit altogether subjects which might have been with propriety introduced, for example, the general theory of *shadows;* and have only touched upon others which would require perhaps further development.

Among other applications of the method, I trust that to the theory of *reciprocal polars* will be found simple and satisfactory.

My attention has been just directed by a friend to a letter from M. Chasles, dated December 10, 1829, published in the *Correspondence Mathématique* of M. Quetelet, tom. vi. p. 81, in which the writer asserts his claim to the invention of a system of coordinates, noticed by M. Plucker in one of the livraisons of Crelle's Journal, to which work I have never had an opportunity of referring. After some preliminary observations, he states his system as follows:—"Pour cela, par trois points fixes A, B, C, je mène trois axes parallèles entre eux, un plan quelconque rencontre ces axes en trois points dont les distances aux points A, B, C, respectivement, sont les coordonnées x, y, z, du plan," &c.; and then goes on to apply his system to a few examples, using the principles and notation of the differential calculus. To any one consulting the letter from which the above extract is taken, it will be apparent that the method there proposed, however excellent and ingenious it may be, bears not the least resemblance to the one developed in the following pages.

Some valuable improvements in the notation I have adopted, have been suggested by the Reverend Charles Graves, F. T. C. D., of which I have thankfully availed myself.

J. B.

TRINITY COLLEGE, DUBLIN,
March 25th, 1840.

ON

TANGENTIAL COORDINATES.

CHAPTER I.

IT must have often appeared an anomalous fact in the application of algebraic analysis to geometrical investigations, that while the locus of a point could be found from the simplest and most elementary considerations, the envelope of a right line or plane could be determined only by the aid of principles, artificial and obscure, derived from a higher department of analysis.

But this is not the only or the greatest objection to the method at present universally followed—it is in most cases operose, and in some impracticable, to reduce the equation $V = 0$ to the form $\frac{dV}{d\alpha} = 0$ and then eliminate the auxiliary variable α, between those equations, a difficulty which becomes far more formidable in problems of three dimensions, where we are obliged to eliminate the auxiliary variables α and β between the three equations

$$V = 0, \quad \frac{dV}{d\alpha} = 0, \quad \frac{dV}{d\beta} = 0.$$

As it follows *à priori* from the principle of *duality*,* that for every locus of a point there exists a corresponding envelope of a right line or plane, it would seem that the comparative paucity of theorems

* See various memoirs on this subject by MM. Gergonne, Poncelet, and others, dispersed through the volumes of the *Annales Mathématiques*.

of the latter species generally known, can be owing to nothing but the want of a simple and direct mode of investigation.

From these considerations I have been led to the discovery of a method simple in principle, and easy of application, analogous to, but different from, that of rectilinear or *projective* coordinates,—as for distinction they may be called,—in which the reciprocals of the distances of the origin from the points where the axes of coordinates are met by a right line, or plane, touching a curve or curved surface, are denoted by the letters ξ, υ, ζ; an equation established between them, may be called the *tangential equation* of the curve or curved surface.

By the help of this equation we may elude the necessity of differentiating the equation $V = 0$, and discover the envelopes of right lines and planes with the same facility as the locus of a point by projective coordinates.

But it is not alone in inquiries of this nature that the method is chiefly valuable; there is a large class of theorems relating to curves touching given right lines, and surfaces in contact with given planes, which may be treated by the method proposed with the greatest facility, whose solution by projective coordinates would lead to exceedingly complicated and unmanageable expressions.

I.

ON THE TANGENTIAL EQUATION OF A POINT IN A PLANE.

Through any point in the plane let two rectangular axes OX, OY, be drawn, let α, β, denote the projective coordinates of the point on those axes, ξ and υ the reciprocals of the intercepts of the axes OX, OY, from the origin made by any line passing through the point, the position of the point is determined by the equation

$$\alpha\xi + \beta\upsilon = 1. \tag{1}$$

The tangential equations of a right line are

$$\xi = \text{constant}, \qquad \upsilon = \text{constant}. \tag{2}$$

II.

ON THE TRANSFORMATION OF COORDINATES.

Let $\frac{1}{\xi}$ and $\frac{1}{\upsilon}$ denote the intercepts of two rectangular axes OX, OY, by a given right line, $\frac{1}{\xi'}$ and $\frac{1}{\upsilon'}$ the intercepts by the same right line of two other rectangular axes OX′, OY′, making an angle θ with the former, then

$$\xi = \xi' \cos\theta + \upsilon' \sin\theta; \qquad \upsilon = \upsilon' \cos\theta - \xi' \sin\theta. \tag{3}$$

The axes remaining parallel, let the origin be translated to a point whose projective coordinates are p and q, let $\frac{1}{\xi'}$ and $\frac{1}{\upsilon'}$ be the intercepts of the new axes; then

$$\xi = \frac{\xi'}{1 + p\,\xi' + q\,\upsilon'}, \qquad \upsilon = \frac{\upsilon'}{1 + p\,\xi' + q\,\upsilon'}; \tag{4}$$

hence $\frac{\upsilon}{\xi} = \frac{\upsilon'}{\xi'}$.

III.

ON THE TANGENTIAL EQUATIONS OF THE CENTRAL CONIC SECTIONS.

The projective equation of a conic section referred to its centre and axes being

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1; \tag{a}$$

the equation of a tangent to it is $\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$. when $y = 0$, $x = \frac{1}{\xi}$, when $x = 0$, $y = \frac{1}{\upsilon}$, $\therefore x' = a^2\xi$, $y' = b^2\upsilon$, substituting in (a) the tangential equation of the section becomes

$$a^2\xi^2 + b^2\upsilon^2 = 1. \tag{5}$$

Let the axes of coordinates now be conceived to revolve through an angle θ round O, and then translated to a point of which the projective coordinates are p and q, by the aid of formulæ (3) and (4) equation (5) is transformed into

$$\left.\begin{array}{l}[a^2\cos^2\theta + b^2\sin^2\theta - p^2]\,\xi^2 + [b^2\cos^2\theta + a^2\sin^2\theta - q^2]\,\upsilon^2 \\ \quad + 2\,[(a^2 - b^2)\sin\theta\cos\theta - p\,q]\,\xi\,\upsilon - 2\,p\,\xi - 2\,q\,\upsilon = 1;\end{array}\right\} \tag{6}$$

hence the tangential equation of a conic section being given in the general form

$$\alpha\,\xi^2 + \alpha'\,\upsilon^2 + 2\beta\,\xi\,\upsilon + 2\gamma\,\xi + 2\gamma'\,\upsilon = 1. \tag{7}$$

Comparing its coefficients with those of (6) we shall have five equations, to determine the projective coordinates ef the centre, the magnitude and inclination of the axes of the section. In the first place $\gamma = -p$, $\gamma' = -q$; hence one half the coefficients of the linear terms in the tangential equation are the projective coordinates of the centre.

Comparing the three remaining coefficients, and introducing the values of p and q we have

$$a^2\cos^2\theta + b^2\sin^2\theta = \alpha + \gamma^2; \quad b^2\cos^2\theta + a^2\sin^2\theta = \alpha'^2 + \gamma'^2;$$

and $(a^2 - b^2)\sin\theta\cos\theta = \beta + \gamma\gamma'$;

hence
$$\tan 2\theta = \frac{2(\beta+\gamma\gamma')}{(\alpha+\gamma^2)-(\alpha'+\gamma'^2)}; \tag{8}$$

$$a^2 = \frac{\alpha+\alpha'+\gamma^2+\gamma'^2 \pm \sqrt{\{(\alpha+\gamma^2)-(\alpha'+\gamma'^2)\}^2+4(\beta+\gamma\gamma')^2}}{2}. \tag{9}$$

The curve is an ellipse or hyperbola according as

$$(\beta+\gamma\gamma')^2 < \text{or} > \text{than } (\alpha+\gamma^2)(\alpha'+\gamma'^2).$$

Let τ = tangent of the angle which one of the asymptots of an hyperbola makes with the axis of x, then

$$\tau = \frac{-(\beta+\gamma\gamma') \pm \sqrt{(\beta+\gamma\gamma')^2-(\alpha+\gamma^2)(\alpha'+\gamma'^2)}}{\alpha}. \tag{10}$$

When the two conditions

$$\alpha+\gamma^2 = \alpha'+\gamma'^2, \text{ and } (\beta+\gamma\gamma') = 0,$$

are satisfied, the curve is a circle, and the origin is at a focus when $\alpha = \alpha'$ and $\beta = 0$.

Now the projective equation of a conic section being

$$A x^2 + A' y^2 + 2B xy + 2C x + 2C' y = 1,$$

it may be shown that the origin is at a focus when $A + C^2 = A' + C'^2$ and $B + CC' = 0$, and the curve is a circle when $A = A'$ and $B = 0$, which conditions are reciprocally analogous to those just mentioned.

The origin of coordinates is on the curve when $\alpha\alpha' - \beta^2 = 0$.

When $\alpha' = 0$ the curve touches the axis of x, when $\alpha = 0$ it touches the axis of y.

Let the general tangential equation of a conic senction

$$\alpha\xi^2 + \alpha' v^2 + 2\beta\xi v + 2\gamma\xi + 2\gamma' v = 1,$$

be solved for v,

$$v = -\frac{(\beta\xi+\gamma')}{\alpha'} \pm \frac{\sqrt{M}}{\alpha'}.$$