THE CONTENTS OF THE FIFTH AND SIXTH BOOKS OF EUCLID

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649462827

The Contents of the Fifth and Sixth Books of Euclid by M. J. M. Hill

Edited by Trieste Publishing Pty Ltd.
Cover @ 2017

www.triestepublishing.com

M. J. M. HILL

THE CONTENTS OF THE FIFTH AND SIXTH BOOKS OF EUCLID

Trieste

THE CONTENTS

OF THE

FIFTH AND SIXTH BOOKS OF EUCLID.

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 50, WELLINGTON STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN COMPANY.
Bombay: E. SEYMOUR HALE.

Euclides

THE CONTENTS

OF THE

FIFTH AND SIXTH BOOKS OF EUCLID

ARRANGED AND EXPLAINED

BY

M. J. M. HILL, M.A., D.Sc., F.R.S.
PROFESSOR OF MATHEMATICS AT UNIVERSITY COLLEGE, LONDON;
LATE EXAMINER IN THE UNIVERSITY OF LONDON,
AND FOR THE CIVIL SERVICE OF INDIA.

CAMBRIDGE:
AT THE UNIVERSITY PRESS.
1900

PREFACE.

THE object of this work is to remove the chief difficulties felt by those who desire to understand the Sixth Book of Euclid. It contains nothing beyond the capacity of those who have mastered the first four Books, and has been prepared for their use. It is the result of an experience of teaching the subject extending over nearly twenty years. The arrangement here adopted has been used by the Author in teaching for the past three years and has been more readily understood than the methods in ordinary use, which he had previously employed.

The Sixth Book depends to a very large extent on the Fifth, but this Fifth Book is so difficult that it is usually entirely omitted with the exception of the Fifth Definition, which is retained not for the purpose of proving all the properties of ratio required in the Sixth Book, but only for demonstrating two important propositions, viz., the 1st and 33rd.

The other properties of ratio required in the Sixth Book are usually assumed, or so-called algebraic demonstrations are supplied. The employment side by side of these two methods of dealing with ratio confuses the learner, because, not being equivalent, they do not constitute, when used in this way, a firm basis for the train of reasoning which he is attempting to follow. A better method is sometimes attempted. This is to insist on the mastering of the Fifth Book, expressed in modern form as in the Syllabus of the Association for the Improvement of Geometrical Teaching, before commencing the Sixth Book.

But it is far too difficult for all but the best pupils, and even they do not grasp the train of reasoning as a whole, though they readily admit the truth of the propositions singly as consequences of the fundamental definitions, which are

(I) The fifth definition, which is the test for the sameness of two ratios.

(II) The seventh definition, which is the test for distinguishing the greater of two unequal ratios from the smaller.

*(III) The tenth definition, which defines "Duplicate Ratio."

*(IV) The definition marked A by Simson, which defines the process for compounding ratios.

In order to make things clear, it is necessary to explain what it is that makes Euclid's Fifth Book so very difficult.

There is first the difficulty arising out of Euclid's notation for magnitudes and numbers. This has been entirely removed in most modern editions by using an algebraic notation and need not therefore be further considered.

There is next the difficulty arising out of Euclid's use of the word "ratio," and the idea represented by it.

His definition of ratio furnishes no satisfactory answer to the question, "What is a ratio?" and it is of such a nature that no indication is afforded of the answer to the still more important question, "How is a ratio to be measured?" As Euclid makes no use of the definition in his argument, it is useless to examine it further, but it is worth while to try to get at his view of ratio. He asserts indirectly that a ratio is a magnitude, because in the seventh definition he states the conditions which must be satisfied in order that one ratio may be *greater* than another. Now the word "greater" can only be applied to a magnitude. Hence Euclid must have considered a ratio to be a magnitude†. To this conclusion it may be objected that if Euclid thought that a ratio was a magnitude he would not so constantly have spoken of the *sameness* of two ratios, but of their *equality*. One can only surmise that, whenever it was possible, he desired to leave open all questions as to the nature of ratio, and to present all his propositions as logical deductions from his fundamental definitions. Yet the question as to the nature of ratio is one which forces itself on the careful reader, and is a source of the greatest perplexity, culminating when he reaches the 11th and 13th Propositions.

The 11th Proposition may be stated thus:—

If	$A : B$ is the same as $C : D$,
and if	$C : D$ is the same as $E : F$,
then	$A : B$ is the same as $E : F$.

* These are not required until the 6th Book is reached.

† Some writers maintain that the word "greater" as applied to ratio, is not used in the same sense as when it is applied to magnitudes. This seems to make matters far more difficult.

Now if a ratio is a magnitude, this only expresses that if $X = Y$, and if $Y = Z$, then $X = Z$.

As this result follows from Euclid's First Axiom it is difficult to see the need for a proof.

This only becomes apparent when the reader realises that Euclid's procedure may be described thus:—

Let A, B, C, D be four magnitudes satisfying the conditions of the Fifth Definition, and let C, D, E, F be four magnitudes also satisfying the same conditions, then it is proved that A, B, E, F also satisfy the conditions of that definition.

Remarks of a somewhat similar nature apply to the 13th Proposition.

In this book it is shewn that two commensurable magnitudes determine a real number; and this real number is called the *measure of their ratio*. The proof of the proposition that two incommensurable magnitudes of the same kind determine a real number (which is taken as the measure of their ratio) is too difficult to find a place in an elementary text-book like this.

A still greater difficulty than the preceding arises from the fact that Euclid furnishes no explanation of the steps by which he reached his fundamental definitions.

To write down a definition, and then draw conclusions from it, is a process which is useful in Advanced Mathematics; but it is wholly unsuitable for elementary teaching. It seems not unlikely that Euclid reached his fundamental definitions as conclusions to elaborate trains of reasoning, but that finding great difficulty in expressing this reasoning in words owing to the absence of an algebraic notation, he preferred to write down his definitions as the basis of his argument, and to present the propositions as logical deductions from his definitions.

Apparently he has left no trace of the steps by which he reached his fundamental definitions; and one of the chief objects of this book is to reconstruct a path which can be followed by beginners from ideas of a simpler order to those on which his work is based.

The most vital of his definitions is the Fifth, on reaching which the beginner, who has read the first four books of Euclid, experiences a sense of discontinuity. He knows nothing which can lead him directly to it, he has no ideas of a simpler order with which to connect it; and he is therefore reduced to learning it by rote.

His teacher may show him that it contains the definition of Proportion given in treatises on Algebra; but even with this assistance it remains difficult for him to remember its details. He may and frequently does learn to apply it correctly in demonstrating the 1st and 33rd Propositions of the Sixth Book, but the Author's experience both of teaching and examining leads him to the belief that it is not really understood.

The explanation here given of the Fifth Definition, apart from the actual notation employed, is that given by De Morgan in his treatise on the Connexion of Number and Magnitude published in the year 1836, and is made clear by a device for exhibiting the order of succession of the multiples of two magnitudes of the same kind, when arranged together in a single series in ascending order of magnitude. This device is called *the relative multiple scale of the two magnitudes.* The notation employed to exhibit it is substantially due to Professor A. E. H. Love, F.R.S. This notation attaches a graphical representation to the Fifth Definition, which appeals to the eye of the learner (See Arts. 29—34).

The seventh definition, as will presently be shewn, is not required.

The tenth definition, which defines Duplicate Ratio, is here based on that marked A by Simson (See Art. 129).

Definition A, which defines the process for Compounding Ratios, is fully explained in four stages, commencing with the general idea on which the process is based, and ending with the proof of the fact that the process employed will always lead to consistent results (See Art. 127).

There remains but one great difficulty for consideration. This is the indirectness of Euclid's line of argument, arising from the fact that he uses the Seventh Definition where the Fifth alone need be employed. His Fifth Definition states the conditions which must be satisfied in order that two ratios may be the same (or if ratios are magnitudes, that they may be equal).

If this definition is a good and sound one, it is evident that it ought to be possible to deduce from it all the properties of equal ratios. This is in fact the case. It is wholly unnecessary to employ the Seventh Definition, which refers to unequal ratios, to prove any of the properties of equal ratios. Its use only renders the proofs of the propositions indirect and artificial and consequently difficult. Not only does no inconvenience result from avoiding its use, but it is possible to get rid of the latter part of the 8th Proposition, and of the whole of the 10th and 13th