

**ELEMENTS OF THE INTEGRAL
CALCULUS, WITH A
KEY TO THE SOLUTION OF
DIFFERENTIAL EQUATIONS**

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649571826

Elements of the Integral Calculus, With a Key to the Solution of Differential Equations by
William Elwood Byerly

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WILLIAM ELWOOD BYERLY

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INTEGRAL CALCULUS,

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EQUATIONS.*

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BOSTON:
PUBLISHED BY GINN, HEATH, & CO.
1881.

PREFACE.

THE following volume is a sequel to my treatise on the Differential Calculus, and, like that, is written as a text-book. The last chapter, however, a Key to the Solution of Differential Equations, may prove of service to working mathematicians.

I have used freely the works of Bertrand, Benjamin Peirce, Todhunter, and Boole; and I am much indebted to Professor J. M. Peirce for criticisms and suggestions.

I refer constantly to my work on the Differential Calculus as Volume I.; and for the sake of convenience I have added Chapter V. of that book, which treats of Integration, as an appendix to the present volume.

W. E. BYERLY.

CAMBRIDGE, 1881.

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