ELEMENTARY THEOREMS RELATING TO DETERMINANTS

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WILLIAM SPOTTISWOODE

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By

WILLIAM SPOTTISWOODE, M.A.,

OF BALLIOL COLLEGE, OXFORD.

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PREFACE.

The variety of problems to which the Theory of Determinants has recently been applied renders it desirable that this branch of analysis should be made generally accessible. But although the principal theorems are familiar to the more advanced mathematicians, there has hitherto been no elementary work upon the subject, to which reference can be readily made by the student.

The Theory is neither lengthy nor intricate, being in fact little else than a method of arrangement, by means of which the results of certain long algebraic processes may be discovered without actually effecting the operations; and indeed, with the exception of a few theorems relating to the addition, multiplication, &c. of determinants, it may be said to consist entirely in its application. Like all similar calculi, it may be carried out into very numerous details; but although this has not been attempted in the present investigations, the principal modifications of form and varieties of combination have been noticed, and the theorems throughout illustrated by examples. The reader will be thereby enabled generally to apply the processes whenever opportunity occurs, and to comprehend any new theorems which may hereafter be proposed. The demonstrations here offered are principally original, although perhaps not different from such as may have occurred to others who have paid attention to the subject.

The functions which are the subject of the present paper, or cases of them more or less general, have for many years been an object of interest to mathematicians; in fact so long ago as the year 1750, Cramer, in his *Introduction à l'Analyse des lignes Courbes* (Appendix), has exhibited the determinants arising from linear equations in the case of two or three variables, and has indicated the law according to which they would be formed in the case of a greater number. In the *Histoire de l'Academie Royale des Sciences, Année* 1764 (published in 1767) Bézout has investigated the degree of the equation resulting from the elimination of unknown quantities from a given system of equations, and has at the same time noticed several cases of determinants, without however entering upon the general law of formation, or the properties of these functions. The *Hist. de l'Académie, An.* 1772, Part II. (published in 1776) contains papers by Laplace and Vandermonde relating to determinants of the second, third, fourth, &c. order. The former, in discussing a system of simultaneous differential equations, has given the law of formation, and shown that when two horizontal or vertical rows (according to the notation of the present work) are interchanged, the sign of the determinant is changed. Vandermonde's paper is upon elimination, and considering the period at which it was written, is remarkable for its elegance; the notation, which is worth noticing, is as follows; the system of quantities being thus represented,

$$\begin{matrix} {}^{1}1 & {}^{1}2 & .. & {}^{1}n \\ {}^{2}1 & {}^{2}2 & .. & {}^{2}n \\ . & . & .. & . \\ {}^{n}1 & {}^{n}2 & .. & {}^{n}n \end{matrix}$$

a determinant of the nth order is written thus,

$$\frac{1 \mid 2 \mid .. \mid n}{1 \mid 2 \mid .. \mid n}$$

so that

$$\frac{1 \mid 2}{1 \mid 2} = {}^{1}1\ {}^{2}2 - {}^{2}1\ {}^{1}2$$

and so on for other orders.

In the *Memoires de l'Académie de Berlin*, 1773, Lagrange has demonstrated that the square of a determinant of the third order is itself a determinant; these formulæ he applies to the establishment of theorems relating to triangular pyramids, and to the problem of the rotation of a solid body. Subsequently to this, Gauss, in his *Disquisitiones Arithmeticæ*, has shown (Section V. Nos. 159 and 270) that the product of two determinants is itself a determinant in the cases of the second and third orders. The whole of this section, which forms a large portion of the work, is devoted to these functions. The case of determinants of the second order arising from quadratic functions of two variables, i.e. of the form $b^2 - ac$, or, adopting his notation, (a, b, c), is very completely discussed. And besides the theorem above noticed, the following problem, which has some connexion with determinants of determinants, is solved: "Given any three whole numbers a, a', a'', (which are not all $= 0$), to find six others, B, B′, B″, C, C′, C″, such that $B'C'' - B''C' = a$, $B''C - BC'' = a'$, $BC' - B'C = a''$." This mathematician appears to have also introduced the term Determinant.

In 1812 Binet published a memoire upon this subject, and established all the principal theorems for determinants of the second, third, and fourth orders; and has further applied his formulæ to the discussion of rhomboids, surfaces of the second order, and properties of solid bodies. See *Journal de l'Ecole Polytechnique, tome* IX. *cahier* 16. The next volume of this series contains a paper by Cauchy, written at the same time, on functions which only change sign when the variables which they contain are transposed. The second part of this paper refers immediately to determinants, and contains a large number of very general theorems. Amongst them is noticed a property of a class of functions closely connected with determinants, first given, so far as I am aware, by Vandermonde; if in the development of the expression

$$a_1 a_2 \cdot\cdot a_n (a_2 - a_1)(a_3 - a_1)(a_n - a_1)(a_3 - a_2) \cdot\cdot (a_n - a_2)(a_n - a_{n-1})$$

the indices be replaced by a second series of suffixes, the result will be the determinant

$$S(\pm a_{1,1} \; a_{2,2} \cdot\cdot a_{n,n})$$

Several papers appeared subsequently from time to time upon various points connected with the subject; but by far the most complete are two by M. Jacobi (Crelle, tom. XXII.) *De formâ et proprietatibus Determinantium*, and *De Determinantibus Functionalibus.* In the same Journal (tom. XXXII. and XXXVIII.) there are two memoires by Mr. Cayley, *Sur les Determinantes Gauches*, which expression has been rendered *Skew Determinants*, the term being adopted from the corresponding translation in Geometry.

References will be found in the course of this work to other papers in which Determinants have been employed, all of which may be consulted with advantage. Besides these, there may be mentioned the following; "On the Theory of Elimination," by Mr. Cayley, *Camb. & Dub. Math. Journal*, vol. III.; "On a new Class of Theorems, &c." by Mr. Sylvester, *Phil. Mag.* vol. XXXVII.; "Extraits de lettres de M. Ch. Hermite, à M. C. G. J. Jacobi, sur differents objects de la théorie des nombres," *Crelle*, tom. XL.

Besides that which is here discussed, there is another very interesting and apparently important theory lately proposed by Mr. Cayley relating to functions, which he calls *Hyperdeterminants.* The general question therein proposed is, "'To find all the derivatives of any number of functions which have the property of preserving their forms unaltered after any linear transformations of the variables.' By derivative is to be understood a function deduced in any manner whatever from the given function, and by hyperdeterminant derivative, or simply hyperdeterminant, those derivatives which have the property above enunciated." Of this nothing has been here said, but those who are desirous of pursuing the subject will find the principles of it laid down in two papers, *Camb. Math. Journal*, vol. IV., and *Camb. and Dub. Math. Journal*, vol. I., or in *Crelle*, tom. XXX.