ELEMENTS OF VECTOR ANALYSIS; ARRANGED FOR THE USE OF STUDENTS IN PHYSICS

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VECTOR ANALYSIS

ARRANGED FOR THE USE OF STUDENTS IN PHYSICS

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ELEMENTS OF VECTOR ANALYSIS.

By J. Willard Gibbs.

[The fundamental principles of the following analysis are such as are familiar under a slightly different form to students of quaternions. The manner in which the subject is developed is somewhat different from that followed in treatises on quaternions, since the object of the writer does not require any use of the conception of the quaternion, being simply to give a suitable notation for those relations between vectors, or between vectors and scalars, which seem most important, and which lend themselves most readily to analytical transformations, and to explain some of these transformations. As a precedent for such a departure from quaternionic usage, Clifford's *Kinematic* may be cited. In this connection, the name of Grassmann may also be mentioned, to whose system the following method attaches itself in some respects more closely than to that of Hamilton.]

CHAPTER I.

CONCERNING THE ALGEBRA OF VECTORS.

Fundamental Notions.

1. *Definition.*—If anything has magnitude and direction, its magnitude and direction taken together constitute what is called a *vector*.

The numerical description of a vector requires three numbers, but nothing prevents us from using a single letter for its symbolical designation. An algebra or analytical method in which a single letter or other expression is used to specify a vector may be called a *vector algebra* or *vector analysis*.

Def.—As distinguished from vectors the real (positive or negative) quantities of ordinary algebra are called *scalars*.*

As it is convenient that the form of the letter should indicate whether a vector or a scalar is denoted, we shall use the small

* The imaginaries of ordinary algebra may be called *biscalars*, and that which corresponds to them in the theory of vectors, *bivectors*. But we shall have no occasion to consider either of these.

Greek letters to denote vectors, and the small English letters to denote scalars. (The three letters, i, j, k, will make an exception, to be mentioned more particularly hereafter. Moreover, π will be used in its usual scalar sense, to denote the ratio of the circumference of a circle to its diameter.)

2. *Def.*—Vectors are said to be *equal* when they are the same both in direction and in magnitude. This equality is denoted by the ordinary sign, as $\alpha=\beta$. The reader will observe that this *vector equation* is the equivalent of three scalar equations.

A vector is said to be equal to zero, when its magnitude is zero. Such vectors may be set equal to one another, irrespectively of any considerations relating to direction.

3. Perhaps the most simple example of a vector is afforded by a directed straight line, as the line drawn from A to B. We may use the notation $\overline{AB}$ to denote this line as a vector, i. e., to denote its length and direction without regard to its position in other respects. The points A and B may be distinguished as the *origin* and the *terminus* of the vector. Since any magnitude may be represented by a length, any vector may be represented by a directed line; and it will often be convenient to use language relating to vectors, which refers to them as thus represented.

Reversal of Direction, Scalar Multiplication and Division.

4. The negative sign (−) reverses the direction of a vector. (Sometimes the sign + may be used to call attention to the fact that the vector has not the negative sign.)

Def.—A vector is said to be *multiplied* or *divided by a scalar* when its magnitude is multiplied or divided by the numerical value of the scalar and its direction is either unchanged or reversed according as the scalar is positive or negative. These operations are represented by the same methods as multiplication and division in algebra, and are to be regarded as substantially identical with them. The terms *scalar multiplication* and *scalar division* are used to denote multiplication and division by scalars, whether the quantity multiplied or divided is a scalar or a vector.

5. *Def.*—A unit vector is a vector of which the magnitude is unity.

Any vector may be regarded as the product of a positive scalar (the magnitude of the vector) and a unit vector.

The notation α_0 may be used to denote the magnitude of the vector α.

Addition and Subtraction of Vectors.

6. *Def.*—The *sum* of the vectors α, β, &c. (written $\alpha+\beta+$ &c.) is the vector found by the following process. Assuming any point A, we determine successively the points B, C, &c., so that $\overline{\text{AB}}=\alpha$, $\overline{\text{BC}}=\beta$, &c. The vector drawn from A to the last point thus determined is the sum required. This is sometimes called the *geometrical* sum, to distinguish it from an *algebraic* sum or an *arithmetical* sum. It is also called the resultant, and α, β, &c., are called the components. When the vectors to be added are all parallel to the same straight line, geometrical addition reduces to algebraic: when they have all the same direction, geometrical addition like algebraic reduces to arithmetical.

It may easily be shown that the value of a sum is not affected by changing the order of two consecutive terms, and therefore that it is not affected by any change in the order of the terms. Again, it is evident from the definition that the value of a sum is not altered by uniting any of its terms in brackets, as $\alpha+[\beta+\gamma]+$&c., which is in effect to substitute the sum of the terms enclosed for the terms themselves among the vectors to be added. In other words, the commutative and associative principles of arithmetical and algebraic addition hold true of geometrical addition.

7. *Def.*—A vector is said to be subtracted when it is added after reversal of direction. This is indicated by the use of the sign $-$ instead of $+$.

8. It is easily shown that the distributive principle of arithmetical and algebraic multiplication applies to the multiplication of sums of vectors by scalars or sums of scalars:—i. e.,

$$
\begin{aligned}
(m+n+\text{\&c.})\ [\alpha+\beta+\text{\&c.}]=\ & m\alpha+n\alpha+\text{\&c.} \\
& +m\beta+n\beta+\text{\&c.} \\
& +\text{\&c.}
\end{aligned}
$$

9. *Vector Equations.*—If we have equations between sums and differences of vectors, we may transpose terms in them, multiply or divide by any scalar, and add or subtract the equations, precisely as in the case of the equations of ordinary algebra. Hence, if we have several such equations containing known and unknown vectors, the processes of elimination and reduction by which the unknown vectors may be expressed in terms of the known are precisely the same, and subject to the same limitations, as if the letters representing vectors represented scalars. This will be evident if we consider that in the multiplications incident to elimination in the supposed scalar equations the multipliers are the coefficients of the unknown quantities, or functions of these coefficients, and that such

multiplications may be applied to the vector equations, since the coefficients are scalars.

10. *Linear relation of four vectors, Coördinates.*—If α, β, and γ are any given vectors not parallel to the same plane, any other vector ρ may be expressed in the form

$$\rho = a\alpha + b\beta + c\gamma.$$

If α, β, and γ are unit vectors, a, b, and c are the ordinary scalar components of ρ parallel to α, β, and γ. If $\rho = \overline{\mathrm{OP}}$, ($\alpha$, β, γ being unit vectors,) a, b, and c are the cartesian coördinates of the point P referred to axes through O parallel to α, β, and γ. When the values of these scalars are given, ρ is said to be given in terms of α, β, and γ. It is generally in this way that the value of a vector is specified, viz., in terms of three known vectors. For such purposes of reference, a system of three mutually perpendicular vectors have certain evident advantages.

11. *Normal systems of unit vectors.*—The letters i, j, k are appropriated to the designation of a *normal system of unit vectors*, i. e., three unit vectors, each of which is at right angles to the other two and determined in direction by them in a perfectly definite manner. We shall always suppose that k is on the side of the i-j plane on which a rotation from i to j (through one right angle) appears counter-clock-wise. In other words, the directions of i, j, and k are to be so determined that if they be turned (remaining rigidly connected with each other) so that i points to the east, and j to the north, k will point upward. When rectangular axes of X, Y, and Z are employed, their directions will be conformed to a similar condition, and i, j, k (when the contrary is not stated) will be supposed parallel to these axes respectively. We may have occasion to use more than one such system of unit vectors, just as we may use more than one system of coördinate axes. In such cases, the different systems may be distinguished by accents or otherwise.

12. *Numerical computation of a geometrical sum.*—If

$$\begin{aligned} \rho &= a\alpha + b\beta + c\gamma, \\ \sigma &= a'\alpha + b'\beta + c'\gamma, \\ &\&c., \end{aligned}$$

then

$$\rho + \sigma + \&c. = (a + a' + \&c.)\alpha + (b + b' + \&c.)\beta + (c + c' + \&c.)\gamma.$$

I. e., the coefficients by which a geometrical sum is expressed in terms of three vectors are the sums of the coefficients by which the separate terms of the geometrical sum are expressed in terms of the same three vectors.

Direct and Skew Products of Vectors.

13. *Def.*—The *direct product* of α and β (written $\alpha.\beta$) is the scalar quantity obtained by multiplying the product of their magnitudes, by the cosine of the angle made by their directions.

14. *Def.*—The *skew product* of α and β (written $\alpha\times\beta$) is a vector function of α and β. Its magnitude is obtained by multiplying the product of the magnitudes of α and β by the sine of the angle made by their directions. Its direction is at right angles to α and β, and on that side of the plane containing α and β (supposed drawn from a common origin), on which a rotation from α to β through an arc of less than 180° appears counter-clock-wise.

The direction of $\alpha\times\beta$ may also be defined as that in which an ordinary screw advances as it turns so as to carry α toward β.

Again, if α be directed toward the east, and β lie in the same horizontal plane and on the north side of α, $\alpha\times\beta$ will be directed upward.

15. It is evident from the preceding definitions that

$$\alpha.\beta=\beta.\alpha, \quad \text{and} \quad \alpha\times\beta=-\beta\times\alpha.$$

16. Moreover,

$$[n\alpha].\beta=\alpha.[n\beta]=n[\alpha.\beta],$$

and

$$[n\alpha]\times\beta=\alpha\times[n\beta]=n[\alpha\times\beta].$$

The brackets may therefore be omitted in such expressions.

17. From the definitions of No. 11 it appears that

$$\begin{gathered} i.i=j.j=k.k=1, \\ i.j=j.i=i.k=k.i=j.k=k.j=0, \\ i\times i=0, \quad j\times j=0, \quad k\times k=0, \\ i\times j=k, \quad j\times k=i, \quad k\times i=j, \\ j\times i=-k, \quad k\times j=-i, \quad i\times k=-j. \end{gathered}$$

18. If we resolve β into two components β' and β'', of which the first is parallel and the second perpendicular to α, we shall have

$$\alpha.\beta=\alpha.\beta' \quad \text{and} \quad \alpha\times\beta=\alpha\times\beta''.$$

19. $\alpha.[\beta+\gamma]=\alpha.\beta+\alpha.\gamma$ and $\alpha\times[\beta+\gamma]=\alpha\times\beta+\alpha\times\gamma.$

To prove this, let $\sigma=\beta+\gamma$, and resolve each of the vectors β, γ, σ into two components, one parallel and the other perpendicular to α. Let these be $\beta', \beta'', \gamma', \gamma'', \sigma', \sigma''$. Then the equations to be proved will reduce by the last section to

$$\alpha.\sigma'=\alpha.\beta'+\alpha.\gamma' \quad \text{and} \quad \alpha\times\sigma''=\alpha\times\beta''+\alpha\times\gamma''.$$

Now since $\sigma=$ ~~$\alpha+\beta$~~ $\beta+\gamma$ we may form a triangle in space, the sides of which shall be β, γ, and σ. Projecting this on a plane perpendicular to α, we obtain a triangle having the sides β'', γ'', and σ'', which affords the relation $\sigma''=\beta''+\gamma''$. If we pass planes perpendicular to α through the vertices of the first triangle, they will give on a line parallel to α segments equal to β', γ', σ'. Thus we obtain the relation $\sigma'=\beta'+\gamma'$. Therefore $\alpha.\sigma'=\alpha.\beta'+\alpha.\gamma'$, since all the cosines involved in these products are equal to unity. Moreover, if α is a unit vector, we shall evidently have $\alpha\times\sigma''=\alpha\times\beta''+\alpha\times\gamma''$, since the effect of the skew multiplication by α upon vectors in a plane perpendicular to α is simply to rotate them all 90° in that plane. But any case may be reduced to this by dividing both sides of the equation to be proved by the magnitude of α. The propositions are therefore proved.

20. Hence,

$$[\alpha+\beta].\gamma=\alpha.\gamma+\beta.\gamma, \qquad [\alpha+\beta]\times\gamma=\alpha\times\gamma+\beta\times\gamma,$$
$$[\alpha+\beta].[\gamma+\delta]=\alpha.\gamma+\alpha.\delta+\beta.\gamma+\beta.\delta,$$
$$[\alpha+\beta]\times[\gamma+\delta]=\alpha\times\gamma+\alpha\times\delta+\beta\times\gamma+\beta\times\delta;$$

and, in general, direct and skew products of sums of vectors may be expanded precisely as the products of sums in algebra, except that in skew products the order of the factors must not be changed without compensation in the sign of the term. If any of the terms in the factors have negative signs, the signs of the expanded product (when there is no change in the order of the factors), will be determined by the same rules as in algebra. It is on account of this analogy with algebraic products that these functions of vectors are called *products* and that other terms relating to multiplication are applied to them.

21. *Numerical calculation of direct and skew products.*—The properties demonstrated in the last two paragraphs (which may be briefly expressed by saying that the operations of direct and skew multiplication are distributive) afford the rule for the numerical calculation of a direct product, or of the components of a skew product, when the rectangular components of the factors are given numerically. In fact,

if $\qquad \alpha=xi+yj+zk$, and $\beta=x'i+y'j+z'k$;

$$\alpha.\beta=xx'+yy'+zz',$$

and $\qquad \alpha\times\beta=(yz'-zy')i+(zx'-xz')j+(xy'-yx')k.$

22. *Representation of the area of a parallelogram by a skew product.*—It will be easily seen that $\alpha\times\beta$ represents in magnitude the area of the parallelogram of which α and β (supposed drawn from a common origin) are the sides, and that it represents in direction the normal to the plane of the parallel-