THE THEORY OF PERMUTABLE FUNCTIONS

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THE THEORY OF PERMUTABLE FUNCTIONS

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LECTURE I

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1. We shall begin with quite elementary and general notions.

First, let us recall the properties of a sum

$$a_1 + a_2 + \ . \ . \ . \ + a_n = \sum_1^n a_i \,.$$

This operation is both associative and commutative, that is,

$$(a+b) + c = a + (b+c)$$

and

$$a + b = b + a \,.$$

Now we can pass from a sum to an integral by a well-known limiting process. For the sake of simplicity, we shall make use of the definition of Riemann: Given a function $f(x)$ which is defined over an interval ab, we subdivide the interval ab into n parts h_1, h_2, h_3, h_4 . . . . . . h_n. Corresponding to every interval h_i we then take some value f_i of $f(x)$ lying between the upper and lower limits of $f(x)$ on h_i, and we form the sum

$$\sum_1^n f_i \, h_i \,.$$

Now suppose we allow $h_1, h_2, h_3, \ldots\ldots h_n$ to become indefinitely small. Then if a unique limit is approached by the sum regardless of the way in which the subdivision of ab is made, we have

$$lim \sum_1^n{}_i f_i h_i = \int_a^b f(x)\,dx\ .$$

Necessary and sufficient conditions for the existence of this limit are well known. In particular, if the function $f(x)$ is continuous over the interval ab or has at most a finite number of discontinuities, the limit and hence also the integral exists.

2. Now let us form the product

$$a \cdot b \cdot c \cdots\cdots\cdots\ .$$

This operation is associative and commutative, that is to say,

$$(ab)\,c = a\,(bc)$$

and

$$ab = ba\ .$$

It is not worth our while to consider the operation which could be obtained from a product by a limiting process such as the one employed in defining an integral. We should be led to logarithmic integration.

3. However, let us consider a limiting process which leads us to something more than these elementary operations.

Let us choose a set of numbers m_{is}, where $i, s = 1, 2, \ldots g$, which may be written in an array

$$\begin{matrix} m_{11} & m_{12} & \cdot & \cdot & \cdot & \cdot & m_{1g} \\ m_{21} & m_{22} & \cdot & \cdot & \cdot & \cdot & m_{2g} \\ & & \cdot & \cdot & \cdot & \cdot & \\ m_{g1} & m_{g2} & \cdot & \cdot & \cdot & \cdot & m_{gg} \end{matrix}$$

and numbers n_{is}, where $i, s = 1, 2, \ldots g$, that is,

$$\begin{matrix} n_{11} & n_{12} & \cdot & \cdot & \cdot & \cdot & n_{1g} \\ n_{21} & n_{22} & \cdot & \cdot & \cdot & \cdot & n_{2g} \\ & & \cdot & \cdot & \cdot & \cdot & \\ n_{g1} & n_{g2} & \cdot & \cdot & \cdot & \cdot & n_{gg} \end{matrix} .$$

We then consider the operation

$$(1) \quad \sum_{1}^{g}{}_{h} \; m_{ih} \; n_{hr}$$

which we shall call *composition of the second type*. This operation is associative, for if we also introduce a set of numbers p_{is}, where $i, s = 1, 2 \quad . \quad . \quad g$, and write the sum

$$\sum_{1}^{g}{}_{h} \sum_{1}^{g}{}_{k} m_{ih}\ n_{hk}\ p_{ks}$$

the expression which we thus obtain is equivalent to either of the forms

$$\sum_{1}^{g}{}_{k} \left(\sum_{1}^{g}{}_{h} m_{ih}\, n_{hk}\right) p_{ks}$$

$$\sum_{1}^{g}{}_{h} m_{ih} \left(\sum_{1}^{g}{}_{k} n_{hk}\, p_{ks}\right)$$

which proves that the associative law is satisfied.

Making use of the notation

$$\sum_{1}^{g}{}_{h} m_{ih}\ n_{hr} = (m, n)_{ir}$$

we shall have

$$((m, n)\ p)_{is} = (m\ (n, p))_{is}$$

which may be written without the parenthesis, thus,

$$(m, n, p)_{is}\ .$$

The commutative law will in general not be satisfied. When it is, the quantities under consideration are called permutable, and we have

$$(m,\, n)_{ir} = (n,\, m)_{ir} \,.$$

We can at once give an example involving permutable quantities. All that is necessary is to consider

$$(m,\, m)_{ir} \text{ which may be written } (m^2)_{ir}$$
$$(m,\, m,\, m)_{ir} \text{ which may be written } (m^3)_{ir}$$

and so on. And it is clear that

$$(m^h,\, m^k)_{ir} = (m^k,\, m^h)_{ir}\,,$$

since the associative law is satisfied.

4. We shall consider also another operation similar to the last, namely

$$(2) \quad \sum_{i+1}^{s-1}{}_h\; m_{ih}\; n_{hs}$$

which will be called *composition of the first type*. The sum (1) previously considered reduces to this one if we suppose that the numbers are zero unless the second subscript is greater than the first. In other words, we have in this case

$$\begin{matrix} 0 & m_{12} & m_{13} & . & . & . & m_{1g} \\ 0 & 0 & m_{23} & . & . & . & m_{2g} \\ & & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & m_{g-1,\,g} \\ 0 & 0 & 0 & . & . & . & 0\,. \end{matrix}$$