AN ARITHMETIC TREATMENT OF SOME PROBLEMS IN ANALYSIS SITUS, A DISSERTATION, PP.343-380

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649264711

An Arithmetic Treatment of Some Problems in Analysis Situs, A Dissertation, pp.343-380 by L. D. Ames

Edited by Trieste Publishing Pty Ltd.
Cover @ 2017

www.triestepublishing.com

L. D. AMES

AN ARITHMETIC TREATMENT OF SOME PROBLEMS IN ANALYSIS SITUS, A DISSERTATION, PP.343-380

Trieste

AN ARITHMETIC TREATMENT OF SOME PROBLEMS IN ANALYSIS SITUS

A DISSERTATION

SUBMITTED TO THE FACULTY OF ARTS AND SCIENCES OF HARVARD UNIVERSITY IN SATISFACTION OF THE REQUIREMENT OF A THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY

L. D. AMES

BALTIMORE
The Lord Baltimore Press
THE FRIEDENWALD COMPANY
1905

*An Arithmetic Treatment of Some Problems in Analysis Situs.**

BY L. D. AMES.

CONTENTS.

INTRODUCTION.

C. Jordan † has proved that the most general simple closed curve divides the plane into an interior and an exterior region. But he assumes all needed facts in regard to polygons without stating clearly just what those assumptions are. He certainly makes use of more than the special case of the same theorem for polygons. Later A. Schoenflies ‡ proved the theorem for a more restricted class of curves including polygons. But his proof is not simple. Complete arithmetization is at least impracticable; the writer must leave to the reader the last details, but these details should be only such as the reader can immediately fill in. Where the line shall be drawn must be left to the judgment

---

* An abstract of the principal results of this paper was presented to the American Mathematical Society at its meeting of December, 1903, and was published in the Bulletin, March, 1904.

† *Cours d'Analyse*, 2d ed., Vol. I., §§96-103, 1893.

‡ Göttinger Nachrichten, Math.-Phys. Kl., 1896, p. 79.

of the individual writer. Schoenflies has left far more for the reader to do than have I in the proof that follows.

There is, however, one point in which Schoenflies' work is open to more serious criticism. He proves that any *straight line* which joins an interior point to an exterior point has a point in common with the curve, and then asserts in the theorem, without further consideration, that it is impossible to pass from an interior point to an exterior point without passing through a point of the curve. This does not follow. In fact it is possible to divide the points of the plane into three assemblages S_1, S_2 and B such that a point of S_1 cannot be joined to a point of S_2 by a straight line having no point in common with B, or by a curve consisting of a finite or infinite number of straight lines, but such that this can be done by other simple curves. And the essential difference between these assemblages S_1 and S_2, on the one hand, and the interior or exterior of a curve, on the other, is a property of the interior and exterior which Schoenflies leaves unmentioned in the theorem or proof, and the omission of which is our final point of criticism: namely, that each is a continuum, that is, if any point is an interior (exterior) point all points in its neighborhood are also.

More recently Ch.-J. de la Valleé Poussin * has published an outline of a proof of the same theorem for the most general simple curve. This work appears much more simple than either of the proofs already mentioned, but it is not arithmetic in form, and it is not easy to see how the arithmetization is to be effected. It can, therefore, be regarded only as a sketch of whatever rigorous proof may be made following its lines.

Since the publication of the abstract of the present paper G. A. Bliss † has proved the theorem for a somewhat more general class of curves than those for which Schoenflies proved it. O. Veblen ‡ has recently published a proof for the most general simple curve. None of these proofs deals with the corresponding theorem in three dimensions.

The present paper assumes the axioms of arithmetic but not those of geometry. It contains a proof of the above mentioned theorem for a class of curves more restricted than those of Jordan, of Valleé Poussin, or of Veblen, but more

---

* *Cours d'Analyse infinitesimal*, Vol. I., (1903), §§300-302.

† *The exterior and interior of a plane curve*, Bulletin of the American Mathematical Society (2), Vol. 10, (1904), p. 398.

‡ *Theory of plane curves in non-metrical analysis situs*, Transactions of the American Mathematical Society, Vol. 6, No. 1, Jan. 1905, p. 83.

general than those of Schoenflies or of Bliss. It goes back to fundamental arithmetic principles, and does not assume the theorem for the polygon. Moreover, it is extended in Part II to the corresponding theorem in three dimensions, and it seems highly probable that it could be extended to more than three dimensions. The proof, both for two and for three dimensions, is based on a conception which I have called the *order of a point* with respect to a curve [or surface]. The order is a point function, uniquely defined and constant in the neighborhood of every point not on the curve [or surface], and undefined and having a finite discontinuity at every point of the curve [or surface]. Its value is always a positive or negative integer or zero.

PART I.

IN TWO DIMENSIONAL SPACE.

I.—FUNDAMENTAL CONCEPTIONS.

1. A *point* is a complex of n real numbers $(a, b \,\ldots\ldots)$. This number n is called the number of *dimensions* in which the point lies. An *assemblage* is any collection, finite or infinite, of such points. In two dimensions the assemblage of all the points is called the *plane*. In the earlier chapters we confine ourselves to two dimensions. The numbers x, y are called the *coordinates* of the point P (x, y); the point $(0, 0)$ is called the *origin*; the assemblage of all points of the type $(x, 0)$ is called the x-axis, etc. Distance, straight lines, circles, squares, and other elementary conceptions are assumed to be defined by their usual analytic expressions without explicit mention.

2. *Transformations.* A *point transformation* is a rule by which the points of an assemblage are individually replaced by the points of an assemblage, in general different. If, whenever a property belongs to one of these assemblages it also belongs to the other, it is said to be *invariant of the transformation.* A *rigid transformation* in two dimensions is defined by relations of the type.

$$x = x' \cos \alpha - y' \sin \alpha + x_0,$$
$$y = x' \sin \alpha + y' \cos \alpha + y_0.$$

We shall assume without explicit mention the simpler facts of invariance. We shall use the expression *change of axes* for brevity to denote a rigid transformation whenever it is desired to emphasize the fact that the essential properties of the assemblage are unchanged. The reasoning involved can generally be stated in something like the following form. An assemblage is given concerning which certain facts are known. The assemblage is transformed into a second assemblage by a transformation with respect to which the given facts are known to be invariant. Certain conclusions are reached in regard to the second assemblage. This is then transformed into the given assemblage by the inverse of the first transformation. The conclusions are known to be invariant of this inverse transformation. They therefore apply to the given assemblage. Unless otherwise specified a change of axes shall be effected by a rigid transformation.

3. *Existence of a Minimum.* The following theorem is well known:*

THEOREM. If S_1 and S_2 are two complete † assemblages of points having no point in common, then the distance of any point of S_1 from any point of S_2 has a positive minimum.

4. *Curves.* A *simple curve* ‡ is an assemblage of points (x, y) which can be paired in a one to one manner with the points of the one dimensional interval $(t_0 \leq t \leq t_1)$ in case the curve is *not closed*, and with the points of the circle

$$\xi = \cos \lambda t, \qquad \eta = \sin \lambda t,$$

in case the curve is *closed;* moreover, when t approaches a limiting value $(\bar{t})$ the point (x, y) shall also approach a limiting point $(\bar{x}, \bar{y})$, and this limiting point shall be the point of the curve which is paired with $\bar{t}$. In the case of the open curve the points corresponding to t_0 and t_1 are called *end points.*

It follows from this definition that a *simple curve* can be represented analytically by equations of the form

$$x = \phi(t), \qquad y = \psi(t), \qquad (t_0 \leq t \leq t_1),$$

where $\phi(t)$ and $\psi(t)$ are single valued continuous functions, and

$$\phi(t) = \phi(t') \text{ and } \psi(t) = \psi(t')$$

* Cf., for example, Jordan, *Cours d'Analyse*, 2d ed., Vol. I, §30, last paragraph.

† Professor Pierpont suggests complete as the English equivalent of abgeschlossen.

‡ Cf. A. Hurwitz, *Verhandlungen des ersten Internationalen Mathematiker-Kongresses*, p. 102.

are not simultaneously satisfied in the case of the open curve when $t \neq t'$, and are not simultaneously satisfied in the case of the closed curve when $t \neq t'$ except

that $$\phi(t_0) = \phi(t_1) \text{ and } \psi(t_0) = \psi(t_1).$$

If the curve is closed, let $\omega = t_1 - t_0$. It is then convenient to extend the definition of the functions $\phi(t)$ and $\psi(t)$ to all values of t by means of the relations

$$\phi(t + n\omega) = \phi(t),\ \psi(t + n\omega) = \psi(t),$$

where n is an integer and ω is defined to be the *primitive period* of the pair of functions. Conversely, every assemblage of points defined by the above equations is a simple curve.

A simple curve is said to be *smooth at a point* if the parameter can be so chosen that the first derivatives $\phi'(t)$ and $\psi'(t)$ exist, are continuous, and do not both vanish at the point. If the point is an end point one sided derivatives are admitted. A *smooth curve* is a simple curve which is smooth at every point. A *regular curve* consists of a chain of smooth curves. Analytically, it is an assemblage which can be defined by the equations

$$x = \phi(t), \qquad y = \psi(t), \qquad (t_0 \leq t \leq t_1),$$

where $\phi(t)$ and $\psi(t)$ are single valued continuous functions whose first derivatives $\phi'(t)$ and $\psi'(t)$ exist, are continuous and do not vanish simultaneously, except possibly at a finite number of exceptional points called *vertices*. Moreover, these derivatives approach limits as the point t approaches any such exceptional value t' from above, and also when t approaches t' from below, and in each case the limits approached by $\phi'(t)$ and $\psi'(t)$ are not both zero; the forward limits are not both equal respectively to the backward limits. It follows that one sided derivatives exist at the exceptional point and that they are equal to the respective limits.

A regular curve may admit *multiple points*, that is, points common to two or more of the constituent smooth curves, other than the common end points of two successive smooth curves. Arithmetically such points correspond to distinct values of t. Two or more of the constituent smooth curves of a regular curve may coincide along whole arcs. Such curves may be treated arithmetically in the same way as the Riemann surface is treated. We do not need such curves,