HINTS FOR THE SOLUTION OF PROBLEMS IN THE THIRD EDITION OF SOLID GEOMETRY

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PERCIVAL FROST

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Trieste

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SOLID GEOMETRY

BY
PERCIVAL FROST, D.Sc., F.R.S.
FORMERLY FELLOW OF ST. JOHN'S COLLEGE,
FELLOW OF KING'S COLLEGE,
MATHEMATICAL LECTURER OF KING'S COLLEGE.

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PREFACE.

THE fulfilment of my promise to give an appendix, containing solutions or hints for the solution of all the problems given in my Third Edition of *Solid Geometry*, has entailed much labour; but this labour will not have been thrown away if it should in any degree have added to the usefulness of the book; at all events it has enabled me to detect many errors and omissions in the statement of the problems which might have given trouble to the student. A table of these errata is given on the following page.

Mr. Chree, Mr. Berry, and Mr. Richmond have shewn no discontinuity in their kindness, for they have not only corrected the proof sheets, but have detected important errors in the problems, as *e.g.* in LX. (7) (Mr. Berry), and in LVIII. (3) (Mr. Richmond); the geometrical solutions of LII. (1) and LXIV. (9) were given by Mr. Berry and Mr. Richmond. I wish to thank Mr. Chree especially for his superintendence of the printing during my absence in the Long Vacation, and I am glad to have this opportunity of noticing a great improvement on the last two lines of my solution of XLIII. (4), which was suggested by him but unfortunately arrived too late, viz. "if (x, y, z, w) be the centre, the left side of (1) $= -R^2$."

ERRATA IN HINTS FOR SOLUTION.

PAGE
44, XXXI. (9), reference to fig. 1 is omitted.
75, L. (2), line 2, *for* $(r\xi^2$, *read* $\frac{1}{2}(r\xi^2$.
77, LII. (6), line 2, *for* $x^{-2}y^{-2}$, *read* $x^{-1}y^{-1}$.
79, LII. (2), line 4, *for* PS, *read* QS.

PROBLEMS.

Errata *majora*.

PAGE

113, XX. (8), *insert* $+ abc = 0$ *after* cxy.

128, XXIII. (9), line 5, *insert* $-$ *before* x.

179, XXIX. (1), line 12, *for* $\sqrt{\tfrac{1}{3}}$, *read* $\sqrt{3}$.

line 18, *for* 1, 0, 1, *read* 1, 0, -1.

180, XXIX. (9), line 3, *insert* $+ a'' \sqrt{a/(a+b)}$ *after* $y\sqrt{b}$.

225, XXXVI. (9), *add* for the same height of the luminous point.

XXXVII. (7), line 2, *dele* double.

226, XXXVIII. (8), *add* a is the intersection of tangent planes at B, C, D.

236, XL. (3), *dele* of revolution.

(7), *insert* $+a'$ *after* $-C'z$).

301, XLIX. (6), *for* $4\pi\{1 - c/\sqrt{(a^2+c^2)}\}$, *read* $4\pi a/\sqrt{(a^2+c^2)}$.

303, LI. (7), line 3, *for* the portion, *read* any portion.

line 4, *for* π, *read* 2π.

line 5, *add* estimated symmetrically with respect to the portion.

(9), line 4, *for* ; also &c., *read* along circular parts of their intersection.

328, LV. (3), *add* and the central circular sections.

(5), *for* conoidal surface, *read* right conoid.

329, LVI. (2), line 6, *for* tangent...at P, *read* generator of the scroll through P.

354, LVII. (7), *for* $\frac{dq}{dp}$, *read* $\left(\frac{dq}{dp}\right)^2$.

LVIII. (3), *add* if p, q be measured along fixed generating lines.

(4), line 6, *for* conicoid, *read* helicoidal surface.

356, LX. (2), line 5, *insert* $-$ *before* $\rho_r\sigma_p\tau_q$.

(6), *for* epicycloid, *read* hypocycloid.

(7), line 6, *insert* $+ i\{\phi(p+iu) - \phi(p-iu)\}$ *after* $f(p-iu)$.

372, LXII. (1), line 6, *for* m^2, *read* m.

389, LXVI. (5), *for* $n-2$, *read* $2(n-2)$.

Errata *minora*.

69, XV. (4), *for* BC, *read* PC.

101, XVIII. (14), line 1, *for* (11), *read* (14).

126, XXI. (10), *for* $b^2 - m^2$, *read* $(b^2 - m^2)^2$.

181, XXXI. (7), line 4, *for* $a'^2b'^2c'^2$, *read* $27a'^2b'^2c'^2$.

224, XXXV. (6), *for* ax, *read* as.

248, XLII. (8), line 5, *add* and abc *after* $a'b'c$.

249, XLIII. (10), *for* pair, *read* pairs.

276, XLVI. (7), *for* β, *read* γ.

302, L. (4), *for* p/z, *read* p/x.

329, LVI. (4), *for* ϕ, *read* ψ.

354, LVIII. (1), *for* square, *read* rectangular.

356, LX. (7), line 2, *for* a, *read* a.

lines 9, 10, 11, *for* index ^{2}, *read* ^{3}.

line 10, *omit* $-$ *before* $\sin sp$.

line 11, *for* $\sin sq$, *read* $\sin sp$.

389, LXVI. (3), line 6, *for* a'^2, *read* a^2.

402, LXVII. (6), *for* $(5a^{-1} + b^{-1})$, *read* $(5a + b)$.

HINTS FOR THE SOLUTION OF PROBLEMS

IN THE THIRD EDITION OF

FROST'S SOLID GEOMETRY.

I.

(1) Two points $(\frac{4}{3}a, \frac{2}{3}a, \pm 2a)$.

(2) Prove that $(x-y)^2=0$, two pairs of coincident points (a, a, a) $(-a, -a, -a)$.

(5) Circle in the plane xy.

II.

(1) (i) Cylinder on a circular base touching Oy. (ii) Traces on zx, zy, parabolas; the section by any plane parallel to xy is a straight line. (iii) Sphere, whose centre is (a, b, c). (iv) Generated by parabolas revolving round Oz; or by circles, centres in Oz, intersecting parabolic traces on xz, yz. (v) Planes $z=\pm h$ cut the surface in straight lines through Oz inclined to the plane zx at an angle $\tan^{-1}(h^2/c^2)$. (vi) Generated by a hyperbola parallel to plane xy. (vii) Generated by an ellipse, one axis constant, the other changing from 0 to ∞. (viii) Trace on plane yz the parabola $y^2=cz$, and on plane $z=h$ an equal parabola with vertex $(h, -h, h)$, generating a parabolic cylinder.

(2) Fig. page 3, (i) $r=a\sin\theta$ gives a circle in plane POM, touching Oz, the same for all values of ϕ. (ii) If a circle in plane xy touch Oy and pass through M, $r=OM$ for all values of θ, giving a circle in plane POM. (iii) $\theta=\frac{1}{2}\pi+\frac{1}{4}\pi\sin 4\phi$ for all values of r, OP makes $\angle\, \frac{1}{4}\pi\sin 4\phi$ below plane xy, and generates a surface cutting xy where $\phi=0, \frac{1}{4}\pi, \frac{1}{2}\pi$, &c.

III.

(1) Art. 23, let l, m, n be the direction-cosines, $l \cos\alpha + m \sin\alpha = 0$, $m \sin\gamma + n \cos\gamma = 0$.

(2) $ll' + mm' + nn' = \frac{1}{2}$, $\therefore (l - l')^2 + \ldots = 1$, $l(l - l') + \ldots = \frac{1}{2}$, $l(l - l') + \ldots = -\frac{1}{2}$.

(3) $\sin^2\alpha + \sin^2(\alpha + 45^\circ) + \sin^2(\alpha + 90^\circ) = 1$, $\therefore \alpha + 45^\circ = 0$. Also $\cos^2\alpha + \cos^2 2\alpha + \cos^2 3\alpha = 1$, $\therefore \cos 2\alpha \cos 3\alpha \cos\alpha = 0$.

(4) Fig. page 44, AE, BE perpendicular to CD. $\cos AEB = (2AE^2 - AB^2)/2AE^2$, and $AE = \frac{1}{2}\sqrt{3}\,AB$.

(5) $2(l^2 + m^2) - (l + m)^2 = n^2 = (l - m)^2$, the direction-cosines are $0, \sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}$, and $\sqrt{\frac{1}{2}}, 0, -\sqrt{\frac{1}{2}}$.

(6) Elementary sector of the circular base = elementary triangle of surface $\times\, a/l$.

(7) The relation is not altered when $-l$ is written for l, or $-m$ for m, or $-n$ for n.

(8) Areas are as $3 : 4 : 5$, and, by Art. 36, $\lambda : \mu : \nu = 3 : 4 : 5$, $\therefore \nu = \sqrt{\frac{1}{2}}$.

IV.

(1) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\therefore \cos(\alpha + \beta)\cos(\alpha - \beta) + \cos^2\gamma = 0$, whatever γ is, $\alpha + \beta$ is least when $\alpha = \beta$, similarly $\alpha = \gamma$, $\therefore 3\cos^2\alpha = 1$, whence 3α, the least possible value of $\alpha + \beta + \gamma$, is found.

(2) Shew that $\lambda l + \mu m + \nu n = 0$, and $\lambda + \mu + \nu = 0$; λ, μ, ν being the direction-cosines.

(3) Shew that $c^2(\alpha l^2 + \beta m^2) + \gamma(al + bm)^2 = 0$, hence that

$$l_1 l_2 : m_1 m_2 : n_1 n_2 = c^2\beta + b^2\gamma : a^2\gamma + c^2\alpha : b^2\alpha + a^2\beta.$$

Condition of parallelism is that of equal roots of the quadratic in l, m.

(4) Similarly.

(5) λ, μ, ν direction-cosines required, $\lambda l + \ldots = \lambda l' + \ldots = \lambda l'' + \ldots = \rho$,

$$\therefore \lambda \times \begin{vmatrix} l, & m, & n \\ l', & m', & n' \\ l'', & m'', & n'' \end{vmatrix} = \rho \begin{vmatrix} 1, & m, & n \\ 1, & m', & n' \\ 1, & m'', & n'' \end{vmatrix}$$

$$\text{and } \lambda : \mu : \nu = \begin{vmatrix} 1, & m, & n \\ 1, & m', & n' \\ 1, & m'', & n'' \end{vmatrix} : \ldots : \ldots.$$

For the second case, shew that

$$l = m'n'' - m''n', \ \therefore \lambda : \mu : \nu = l + l' + l'' : \ldots : \ldots.$$

(6) Art. 26, $u + v + w = 2a\beta\gamma + 2b\gamma\alpha + 2c\alpha\beta$, $P = a^2\alpha^2 + \ldots - 2bc\beta\gamma \ldots$.

(7) l, m, n and l', m', n' direction cosines of the lines. Prove $l-l'=-(m-m')$ and $l+l'=m+m'$, $\therefore$ $l=m'$, $m=l'$, $2lm=-n^2$, $(l-m)^2=3n^2$, then $l=\frac{1}{2}(-1\pm\sqrt{\frac{1}{3}})$, $m=\frac{1}{2}(1\pm\sqrt{\frac{1}{3}})$, $n=\mp\sqrt{\frac{1}{3}}$.

(8) Art. 25, λ', μ', ν' required direction cosines,

$$\lambda+\lambda'=2\sqrt{\tfrac{1}{3}}\,(\lambda+\mu+\nu)\sqrt{\tfrac{1}{3}}.$$

(9) $1-\frac{1}{2}(\delta\theta)^2+\ldots=l(l+\delta l)+\ldots$ and $1=(l+\delta l)^2+\ldots$.

(10) Art. 36, a an edge of the cube, $\Sigma A_1=a^2=\Sigma A_3=\Sigma A_6$, normal to plane of maximum projection has equal direction cosines and maximum area $=a^2\sqrt{3}$.

(11) Let θ be the inclination of the planes; the perpendicular from D' on the plane $ABC=DD'\cos\theta$.

(12) Art. 28, L, M, N direction-ratios,

$$-l+L+M\cos\nu+N\cos\mu=0,$$

and three other equations; eliminate L, M and N.

V.

(1) $x=z=-\frac{1}{2}y$, $\cos\alpha=\cos\gamma=-\frac{1}{2}\cos\beta$, $\sec\beta=-\sqrt{\frac{3}{2}}$, $\sec 2\beta=3$.

(2) Use the three equations

$$(x-1)(y-1)(x-y)=0,\quad (y-1)(z-1)(y-z)=0,$$
$$(z-1)(x-1)(z-x)=0,$$

satisfied by $x=y=1$, $z=1$, $y=z$, &c, and $x=y=z$; straight lines passing through $(1, 1, 1)$.

(3) Satisfied by $x=y=z$, direction cosines $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{3}}$, $\sqrt{\frac{1}{3}}$.

(4) $x^2+2xz=0$, $\therefore$ straight lines are $x=0$, $y=0$ and $x=y=-2z$.

(5) Straight line is $(x-b)/(c-b)=(y-c)/(a-c)=(z-a)/(b-a)$ perpendicular to straight line $2x/(b+c)=2y/(c+a)=2z/(a+b)$ and the other two lines.

(6) It is the distance from the origin to the projection of the line on the plane xy, and is $(am\sim bl)/\sqrt{(l^2+m^2)}$. The equations are $lx+my=0$ and $z=\gamma$, which meets the given line, shew that $\gamma(l^2+m^2)+n(al+bm)=0$.

(7) Line joining centres of the edge and diagonal is the shortest distance $=a/\sqrt{2}$.

(8) Take $y=A+Bz/c$ and $mx=A'+B'z/c$ for the intersecting line. Shew that $A'=B$ and $B'=A$, and make $A=m\lambda\sin\theta$, $B=\lambda\cos\theta$.

(9) The points are $(a\cos\alpha, a\sin\alpha, c)$; $(\pm b\cos\alpha, \mp b\sin\alpha, -c)$.

(10) Art. 59, cylinder of evanescent radius. Equation may be written $(ny-mz)^2+(lz-nx)^2+(mx-ly)^2=0$.

(11) Art. 64, for the locus, $z=\frac{1}{2}(c-c)=0$.