AN INTRODUCTION TO THE SUMMATION OF DIFFERENCES OF A FUNCTION, PP. 2-43; SEVEN LESSONS IN THEORY OF INVERSIOIS OF ORDER AND DETERMINANTS, PP.6-32

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Trieste

Compliments of
Benjamin F. Groat.

AN INTRODUCTION

TO THE

SUMMATION OF DIFFERENCES OF A FUNCTION

AN ELEMENTARY EXPOSITION OF THE NATURE
OF THE ALGEBRAIC PROCESSES REPLACED
BY THE ABBREVIATIONS OF THE
INFINITESIMAL CALCULUS

BY

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PREFACE

The subject-matter for the following pages has its origin in various sources well known to the general mathematical reader; yet it is felt that the view of the differential and integral calculus presented in the sequel generally does not come to be recognized by the student sufficiently early in his course. The view referred to is suggested in a number of excellent works upon infinitesimal calculus, but the writer's aim is to treat the subject of this paper as a chapter in algebra, preparatory and supplementary to a course in differential and integral calculus, and not as a part of that course. The historical order of development has been followed in general outline, though the exercises are set in modern notation. The historical order of development of a science is usually (not always; cf. Hist. logarithms) the easiest to follow; moreover any other method withholds history and often fails to give a *sufficient* account of the science involved. This stands to reason for the actual development is likely to be the result of *necessity*. The reader is referred to several excellent articles and books for the history, which, after reading this paper, should be intelligible to the beginner.

The object is not merely to gain the historical view of the infinitesimal analysis but to prepare the student for the solution of problems in applied mathematics. The processes of differentiation and integration are acquired without much difficulty; but to *see the integral* with facility in a problem in analytic mechanics or physics requires clear notions as to sums and limits of sums. Such notions are of much more importance to the physicist and engineer than the more elaborate methods for integrating complicated forms: it is the desire to aid the student in forming these notions early, together with the writer's need of a suitable exercise book for use in his classes, that has been the reason for writing this paper.

The following is suggested as a course in fundamental principles and exercises: Elementary algebra, including progressions; convergence and divergence of series including the elementary test theorems; sums of squares and cubes of first n integers; undetermined coefficients and decomposition of fractions; exponential and logarithmic series and logarithms; elements of trigonometry; summation of series as in Chapter I, this book; derived functions; theory of equations, graphs and elementary limit theorems as stated in Arts. 7, 8, 9 and Theorem I, Art. 14, McMahon and Snyder's *Differential Calculus*; permutations, combinations, binomial theorem; determinants, system of linear equations, elimination, Sylvester's Method, discriminants; analytic geometry of plane and space; Chapter II, this book; differential calculus proper, that is, the rules and formulas of differentiation; Chapters III and IV; integral calculus followed by more complete courses.

For a simple demonstration of the logarithmic series the following is suggested: $e^{y \log(1+x)} = (1+x)^y$; expanding by the exponential and binomial theorems and equating the coefficients of y in the two expansions we obtain the logarithmic series. Of course the limitations of this proof should be noticed. The formulas for the sums of squares and cubes may be proved by induction. An early introduction to the factor and remainder theorems with their application in drawing graphs and locating roots of rational functions is advocated.

The kindness of Mr. H. H. Dalaker in reading proofs and verifying examples is acknowledged.

University of Minnesota
March, 1902

SUMMATION OF DIFFERENCES

CHAPTER I

SUMMATION

1. The Symbol Σ. The series with which we shall have to deal are such that the nth term can be expressed as a function of n. Arithmetical and geometrical series are of this kind. Thus the nth term of the arithmetical series, $a+(a+d)+(a+2d)+\cdots$, is $(a+\overline{n-1}\,d)$; any term may be written from this by substituting the corresponding value of n. As an example the third term of the series is $(a+\overline{3-1}\,d)=(a+2d)$, as it should be. In the geometric series $a+ar+ar^2+\cdots$, the nth term is ar^{n-1}, and from this, for example, the fifth term is $ar^{5-1}=ar^4$, as it should be. Hence we see that the terms of an arithmetic progression are of the type $(a+\overline{x-1}\,d)$, and the terms of a geometric progression of the type ar^{x-1}, where x is to have the particular value corresponding to the number of the term in question.

2. The symbol $\sum$ is employed for the purpose of indicating that a sum of terms is to be taken. We define

$$\sum (a+\overline{x-1}\,d)$$

to mean the sum of terms of the type $(a+\overline{x-1}\,d)$,

$$\sum ar^{x-1}$$

to mean the sum of terms of the type ar^{x-1}, and in general

$$\sum \phi(x)$$

to mean the sum of terms of the type $\phi(x)$. Our definition, however, is not complete, since nothing in the notation indicates how many terms of a given type are to be included in the sum. Suppose, for example, we wish to indicate the sum of the first five terms of the series 3, 5, 7, 9, $\cdots$; we should write simply $\sum(3+2\overline{x-1})$, and have to explain that we are to sum the five values found by making x successively equal to 1, 2, 3, 4, 5, in $(3+2\overline{x-1})$. We complete the definition by attaching the *limits* of x in the following manner:

$$\sum_{x=1}^{x=5}(3+2\overline{x-1});$$

the limits designating the first and last values to be given to x. The limits may be omitted in any problem provided they are understood.

3. The student will have no difficulty in verifying the following illustrative examples:

$$\sum_3^7 2(x+1) = 2\cdot 4 + 2\cdot 5 + 2\cdot 6 + 2\cdot 7 + 2\cdot 8 = \frac{8+16}{2}\cdot 5 = 60.$$

$$\sum_3^7 2^{x-1} = 2^2 + 2^3 + 2^4 + 2^5 + 2^6 = \frac{2^2(2^5-1)}{2-1} = \frac{4\cdot 31}{1} = 124.$$

$$\sum_1^7 x = 1 + 2 + \cdots + 7 = \frac{1+7}{2}7 = 28.$$

$$\sum_1^{10} x^2 = 1^2 + 2^2 + \cdots + 10^2 = \frac{10(10+1)(20+1)}{6} = \frac{10\cdot 11\cdot 21}{6}$$
$$= 5\cdot 11\cdot 7 = 385.$$

$$\sum_1^7 x^3 = \left\{\sum_1^7 x\right\}^2 = \left\{7\frac{1+7}{2}\right\}^2 = 784.$$

$$\sum_3^5\left(x^2+\frac{1}{x}\right) = \sum_3^5 x^2 + \sum_3^5\frac{1}{x} = 3^2 + 4^2 + 5^2 + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$
$$= 50 + \frac{47}{60} = 50\tfrac{47}{60}.$$

$$\sum_{z=3}^{z=5} \log z = \log 60.$$

$$\sum_{3}^{6} (-1)^{z-1} \log z = \log \frac{5}{8}.$$

$$\sum_{n=5}^{10} (-1)^{n-1} \frac{1}{n} = -\sum_{5}^{10} (-1)^{n} \frac{1}{n} = \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10}.$$

$$\sum_{m}^{n} \psi(x) \equiv \psi(m) + \psi(m+1) + \cdots + \psi(n-1) + \psi(n).$$

$$\sum_{z=0}^{3} \sin\left(\frac{z\pi}{2} + x\right)\cos\left(\frac{z\pi}{2} - x\right) = \frac{1}{2}\sum \sin z\pi + \frac{1}{2}\sum \sin 2x = 2 \sin 2x.$$

4. In the preceding paragraph we were concerned with finding the value of $\sum\limits_{m}^{n} \phi(x)$; in the present and following article we shall solve the inverse problem, having given a series of the form $\phi(m) + \phi(m+1) + \cdots + \phi(n)$ to find the equivalent expression $\sum\limits_{m}^{n} \phi(x)$. As an example take the series

$$\frac{3}{4 \cdot 5} - \frac{4}{5 \cdot 7} + \frac{5}{6 \cdot 9} - \frac{6}{7 \cdot 11} + \cdots - \frac{24}{25 \cdot 47}.$$

We observe at once that the signs of the even numbered terms are negative; this is effected in the summation by introducing the *sign factor* $(-1)^{z-1}$, so that when z is odd the sign is +. We notice further that each numerator is greater by 2 than the number of the term in which it stands, while the first factor of the denominator is 3 greater. The last factor of the denominator increases 2 units per term, and is 5 in the first term, hence it is $3+2n$ in the nth term; that is, it exceeds twice the number of the term by 3. The nth term, therefore, expressed as a function of n, may be written

$$(-1)^{n-1} \frac{(n+2)}{(n+3)(2n+3)},$$

and, all together, there are 22 terms.

$$\therefore \frac{3}{4 \cdot 5} - \frac{4}{5 \cdot 7} + \cdots - \frac{24}{25 \cdot 47} = \sum_{x=1}^{x=22} \frac{(-1)^{x-1}(x+2)}{(x+3)(2x+3)}.$$