APPLICATION OF THE ANGULAR ANALYSIS TO THE SOLUTION OF INDETERMINATE PROBLEMS OF THE SECOND DEGREE

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BY C. GILL,

PROFESSOR OF MATHEMATICS IN ST. PAUL'S COLLEGE.

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PREFACE.

The notation now so universally used for the sine, cosine, &c. of an angle, although in some respects inconvenient, inasmuch as it requires several symbols to designate a quantity where one might be sufficient, is yet so admirably adapted to the wants of Analysis, that it has forced its way into every department of the higher Mathematics. It owes this preference, partly to the power it gives of combining the functions of different angles, or of multiple angles; but principally to its enabling us to avoid the use of the radical sign of the second degree; inasmuch as, if the sine of an angle were represented by one symbol, its cosine would be an irrational function of that symbol.

It would seem strange, then, that this notation has never yet been applied to the Diophantine Analysis, the avowed object of which is to render rational algebraic expressions of certain forms. Perhaps one reason for this may be, that, notwithstanding the attention which has been paid to this branch of Analysis by many mathematicians of the highest rank, the object has been confined to the finding of *numbers* that may fulfil certain conditions, rather than *algebraic forms*, to fulfil those conditions. The methods of solution have been more or less tentative in their character, and could therefore scarcely be expected to produce results to be compared with the lofty objects of modern Analysis.

In the few pages which follow, I have attempted not only to apply the notation of Trigonometry to several well known Indeterminate Problems of the second degree, but also to introduce somewhat more of system into the method of investigation, and more of generality into the results of the Analysis than has been yet attained. How far I have succeeded, must be left to the judgment of the reader.

In the two last Problems of the second Chapter, I have applied the Analysis to inquiries which would scarcely be attempted without some better method of notation than that hitherto used.

College Point, N. Y.
March 27th, 1848.

ANGULAR ANALYSIS

APPLIED TO

INDETERMINATE EQUATIONS

OF THE SECOND DEGREE.

CHAPTER I.

NOTATION.

The six trigonometrical functions of an angle A, or of an arc which measures that angle in a circle whose radius is unity, are related to each other as in the following equations:

$$\sin^2 A + \cos^2 A = 1, \qquad \tan A = \frac{\sin A}{\cos A},$$

$$\sin A \operatorname{cosec} A = \cos A \sec A = \tan A \cot A = 1.$$

If, then, the sine and cosine of the angle can be expressed in rational numbers, all the other functions can be so likewise. To do this it is only necessary to take

$$\sin A = \frac{2mn}{m^2+n^2}, \qquad \cos A = \frac{m^2-n^2}{m^2+n^2},$$

which satisfy the first of the above equations, m and n being any rational numbers whatever; then

$$\tan A = \frac{2mn}{m^2 - n^2}, \qquad \cot A = \frac{m^2 - n^2}{2mn},$$

$$\operatorname{cosec} A = \frac{m^2 + n^2}{2mn}, \qquad \sec A = \frac{m^2 + n^2}{m^2 - n^2}.$$

These six equations may all be included in the symbolical form

$$A = \varphi\left(\frac{m}{n}\right), \quad . \quad . \quad . \quad (1)$$

or, in its inverse form

$$\frac{m}{n} = \varphi^{-1}(A); \quad . \quad . \quad . \quad (2)$$

in which, A is any angle or arc,

φ is a functional characteristic,

$\frac{m}{n}$ is the *root of the function.*

For example, when we say that

$$A = \varphi(2),$$

we mean that

$$\sin A = \frac{2 \,.\, 2}{2^2 + 1^2} = \tfrac{4}{5}, \quad \cos A = \frac{2^2 - 1^2}{2^2 + 1^2} = \tfrac{3}{5}, \text{ \&c.}$$

PROBLEM I. To find the root of a function, in terms of the angle.

Solution. Take the equation

$$\frac{m^2 - n^2}{m^2 + n^2} = \cos A,$$

and solve it for $\frac{m}{n}$, we shall find

$$\frac{m^2}{n^2} = \frac{1 + \cos A}{1 - \cos A} = \cot^2 \tfrac{1}{2} A,$$

$$\frac{m}{n} = \cot \tfrac{1}{2} A = \varphi^{-1}(A), \quad . \quad . \quad (3)$$

so that, ***the root of a function is the cotangent of half the angle.***

Cor. While $A = \varphi\left(\frac{m}{n}\right)$ varies from $0°$ to $180°$,

$$\cot \tfrac{1}{2}A = \frac{m}{n} \text{ varies from } \infty \text{ to } 0,$$

passing through *unity* when $A = 90°$;

thus:

$$0° = \varphi\left(\tfrac{1}{0}\right), \quad 90° = \varphi(1), \quad 180° = \varphi(0), \text{ \&c.}$$

hence also;

$$\text{when } \frac{m}{n} > 1, \quad \cdot \quad \varphi\left(\frac{m}{n}\right) < 90° > 0°;$$

$$\text{when } \frac{m}{n} < 1 > 0, \quad \varphi\left(\frac{m}{n}\right) > 90° < 180°.$$

Again, while $A = \varphi\left(\frac{m}{n}\right)$ varies from $180°$ to $360°$,

$$\cot \tfrac{1}{2}A = \frac{m}{n} \text{ varies from } 0 \text{ to } -\infty,$$

passing through *negative unity* when $A = 270°$; thus:

$$270° = \varphi(-1), \quad 360° = \varphi\left(\tfrac{-1}{0}\right), \text{ \&c.};$$

hence also:

$$\text{when } \frac{m}{n} < 0 > -1, \quad \varphi\left(\frac{m}{n}\right) > 180^\circ < 270^\circ;$$

$$\text{when } \frac{m}{n} < -1 \quad , \quad \varphi\left(\frac{m}{n}\right) > 270^\circ < 360^\circ,$$

&c. &c.

PROBLEM II. Having given the roots of two functions, to find the root of their sum.

Solution. Let $\quad A + B = C;$

or if $\quad A = \varphi\left(\frac{m}{n}\right), \; B = \varphi\left(\frac{p}{q}\right), \; C = \varphi\left(\frac{r}{s}\right),$

$$\varphi\left(\frac{m}{n}\right) + \varphi\left(\frac{p}{q}\right) = \varphi\left(\frac{r}{s}\right).$$

But we have

$$\cot\tfrac{1}{2}C = \cot\tfrac{1}{2}(A+B) = \frac{\cot\frac{1}{2}A\cot\frac{1}{2}B - 1}{\cot\frac{1}{2}A + \cot\frac{1}{2}B},$$

or,

$$\frac{r}{s} = \frac{\frac{m}{n}\cdot\frac{p}{q} - 1}{\frac{m}{n} + \frac{p}{q}} = \frac{mp - nq}{mq + np};$$

and the above equation becomes

$$\varphi\left(\frac{m}{n}\right) + \varphi\left(\frac{p}{q}\right) = \varphi\left(\frac{mp - nq}{mq + np}\right). \quad . \quad . \quad (4)$$

Example 1. $90^\circ + \varphi\left(\frac{m}{n}\right) = \varphi(1) + \varphi\left(\frac{m}{n}\right) = \varphi\left(\frac{m-n}{m+n}\right);$ (5)

thus:

$$90^\circ + \varphi(2) = \varphi(\tfrac{1}{3}),$$

$$90^\circ + \varphi(\tfrac{3}{2}) = \varphi(\tfrac{1}{5}),$$

&c.