INTRODUCTION TO INFINITE SERIES

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649334605

Introduction to Infinite Series by William F. Osgood

Edited by Trieste Publishing Pty Ltd.
Cover @ 2017

www.triestepublishing.com

WILLIAM F. OSGOOD

INTRODUCTION TO INFINITE SERIES

Trieste

INTRODUCTION

TO

INFINITE SERIES

BY

WILLIAM F. OSGOOD, PH.D.
ASSISTANT PROFESSOR OF MATHEMATICS IN HARVARD UNIVERSITY

CAMBRIDGE
Published by Harvard University
1897

PREFACE.

IN an introductory course on the Differential and Integral Calculus the subject of Infinite Series forms an important topic. The presentation of this subject should have in view first to make the beginner acquainted with the nature and use of infinite series and secondly to introduce him to the theory of these series in such a way that he sees at each step precisely what the question at issue is and never enters on the proof of a theorem till he feels that the theorem actually requires proof. Aids to the attainment of these ends are: (*a*) a variety of illustrations, taken from the cases that actually arise in practice, of the application of series to computation both in pure and applied mathematics; (*b*) a full and careful exposition of the meaning and scope of the more difficult theorems; (*c*) the use of diagrams and graphical illustrations in the proofs.

The pamphlet that follows is designed to give a presentation of the kind here indicated. The references are to Byerly's *Differential Calculus*, *Integral Calculus*, and *Problems in Differential Calculus*; and to B. O. Peirce's *Short Table of Integrals*; all published by Ginn & Co., Boston.

WM. F. OSGOOD.

CAMBRIDGE, April 1897.

INTRODUCTION.

1. *Example.* — Consider the successive values of the variable

$$s_n = 1 + r + r^2 + \cdot \cdot \cdot \cdot \cdot \cdot + r^{n-1}$$

for $n = 1, 2, 3, \cdot \cdot \cdot \cdot \cdot$ Let r have the value $\frac{1}{2}$. Then

$$\begin{aligned} s_1 &= 1 &&= 1 \\ s_2 &= 1 + \tfrac{1}{2} &&= 1\tfrac{1}{2} \\ s_3 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} &&= 1\tfrac{3}{4} \\ s_4 &= 1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} &&= 1\tfrac{7}{8} \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

If the values be represented by points on a line, it is easy to see the

FIG. 1.

law by which any s_n can be obtained from its predecessor, s_{n-1}, namely: s_n lies half way between s_{n-1} and 2.

Hence it appears that when n increases without limit,

$$\text{Lim } s_n = 2.$$

The same result could have been obtained arithmetically from the formula for the sum s_n of the first n terms of the geometric series

$$a + ar + ar^2 + \cdot \cdot \cdot \cdot \cdot \cdot + ar^{n-1},$$

$$s_n = \frac{a(1 - r^n)}{1 - r}.$$

Here $a = 1$, $r = \frac{1}{2}$,

$$s_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

When n increases without limit, $\frac{1}{2^{n-1}}$ approaches 0 as its limit, and hence as before $\text{Lim } s_n = 2$.

2. *Definition of an Infinite Series.* Let $u_0, u_1, u_2, \cdots\cdots$ be any set of values, positive or negative or both, and form the series

$$u_0 + u_1 + u_2 + \cdots\cdots \qquad (1)$$

Denote the sum of the first n terms by s_n:

$$s_n = u_0 + u_1 + \cdots\cdots + u_{n-1}.$$

Allow n to increase without limit. Then either *a*) s_n will approach a limit U:

$$\lim_{n=\infty} s_n = U;$$

or *b*) s_n approaches no limit. In either case we speak of (1) as an *Infinite Series*, because n is allowed to increase without limit. In case *a*) the infinite series is said to be *convergent* and to have the *value** U or *converge towards* the value U. In case *b*) the infinite series is said to be divergent.

The geometric series above considered is an example of a convergent series.

$$1 + 2 + 3 + \cdots\cdots,$$
$$1 - 1 + 1 - \cdots\cdots$$

are examples of divergent series. Only convergent series are of use in practice.

The notation

$$u_0 + u_1 + \cdots\cdots \textit{ ad inf.} \text{ (or } \textit{to infinity}\text{)}$$

is often used for the limit U, or simply

$$U = u_0 + u_1 + \cdots\cdots$$

* U is often called the *sum* of the series. But the student must not forget that U is *not* a sum, but is the *limit* of a sum. Similarly the expression "the sum of an infinite number of terms" means the *limit* of the sum of n of these terms, as n increases without limit.

I. CONVERGENCE.

a) SERIES, ALL OF WHOSE TERMS ARE POSITIVE.

3. *Example.* Let it be required to test the convergence of the series

$$1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdot\cdot\cdot\cdot\cdot + \frac{1}{n!} + \cdot\cdot\cdot\cdot\cdot \qquad (2)$$

where $n!$ means $1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot\cdot\cdot n$ and is read "*factorial n*". Discarding for the moment the first term, compare the sum of the next n terms

$$\sigma_n = 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdot\cdot\cdot\cdot\cdot + \frac{1}{1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot\cdot\cdot n}$$

with the corresponding sum

$$S_n = 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \cdot\cdot\cdot\cdot\cdot + \frac{1}{\underbrace{2\cdot 2\cdot 2\cdot\cdot\cdot\cdot\cdot\cdot\cdot 2}_{n-1 \text{ factors}}}$$

$$= 2 - \frac{1}{2^{n-1}} < 2 \quad (\text{Cf. § 1}).$$

Each term of σ_n after the first two is less than the corresponding term in S_n, and hence the sum

$$\sigma_n < S_n < 2,$$

or, inserting the discarded term and denoting the sum of the first n terms of the given series by s_n,

$$s_{n+1} = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \cdot\cdot\cdot\cdot\cdot + \frac{1}{1\cdot 2\cdot 3\cdot\cdot\cdot\cdot\cdot n} < 3,$$

no matter how large n be taken. That is to say, s_n is a variable that always increases as n increases, but that never attains so large a value as 3. To make these relations clear to the eye, plot the successive values of s_n as points on a line.