# THE INTEGRAL CALCULUS ON THE INTEGRATION OF THE POWERS OF TRANSCENDENTAL FUNCTIONS, NEW METHODS AND THEOREMS, CALCULATION OF THE BERNOULLIAN NUMBERS, RECTIFICATION OF THE LOGARITHMIC CURVE, INTEGRATION OF LOGARITHMIC BINOMIALS ETC.

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The integral calculus on the integration of the powers of transcendental functions, new methods and theorems, calculation of the Bernoullian numbers, rectification of the logarithmic curve, integration of logarithmic binomials etc. by James Ballantyne



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# JAMES BALLANTYNE

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# THE INTEGRAL CALCULUS

## SECTION ONE

### On the Integration of the Powers of Trigonometrical Functions

**1.** In performing the operation of integrating a differential it is usual to add the constant $C$ to the result, because the integral from which the differential has been derived may contain some constant quantity not affected by the variable; in which case the constant would not be indicated in the differential. In the following Tables and Series the constant has been fully accounted for, and the complete integrals are therein expressed, excepting in those cases where the constant is particularly noted.

TABLE I

**Integrals of Even Powers of $\sin x \cdot dx$, to Radius 1**

$\int \sin^0 x \cdot dx = x.$

$$\int \sin^2 x \cdot dx = \frac{x - \cos x \cdot \sin x}{2}.$$

$$\int \sin^4 x \cdot dx = \frac{3x - 3 \cos x \cdot \sin x - 2 \cos x \cdot \sin^3 x}{2 \cdot 4}.$$

$$\int \sin^6 x \cdot dx = \frac{15\,x - 15 \cos x \cdot \sin x - 10 \cos x \cdot \sin^3 x - 8 \cos x \cdot \sin^5 x}{2 \cdot 4 \cdot 6}.$$

$$\int \sin^8 x \cdot dx = (105\,x - 105 \cos x \cdot \sin x - 70 \cos x \cdot \sin^3 x - 56 \cos x \cdot \sin^5 x - 48 \cos x \cdot \sin^7 x) \div 2 \cdot 4 \cdot 6 \cdot 8$$

$$\int \sin^{10} x \cdot dx = (945\,x - 945 \cos x \cdot \sin x - 630 \cos x \cdot \sin^3 x - 504 \cos x \cdot \sin^5 x - 432 \cos x \cdot \sin^7 x - 384 \cos x \cdot \sin^9 x) \div 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10$$

**2.** This table may be carried to any extent by observing the following law of progression:

$$\int \sin^m x \cdot dx = \frac{m-1}{m} \int \sin^{m-2} x \cdot dx - \frac{1}{m} \cos x \cdot \sin^{m-1} x,$$

and the table may be extended to negative values of $m$, by changing this expression to

$$\int \sin^{m-2}x\cdot dx = \frac{m}{m-1}\int \sin^m x\cdot dx + \frac{1}{m-1}\cos x\cdot\sin^{m-1}x;\ \text{thus,}$$

$$\int \sin^{-2}x\cdot dx = -\cot x + C.$$

$$\int \sin^{-4}x\cdot dx = -\frac{2\cot x + \dfrac{\cot x}{\sin^2 x}}{3} + C.$$

$$\int \sin^{-6}x\cdot dx = -\frac{8\cot x + \dfrac{4\cot x}{\sin^2 x} + \dfrac{3\cot x}{\sin^4 x}}{3\cdot 5} + C.$$

$$\int \sin^{-8}x\cdot dx = -\frac{48\cot x + \dfrac{24\cot x}{\sin^2 x} + \dfrac{18\cot x}{\sin^4 x} + \dfrac{15\cot x}{\sin^6 x}}{3\cdot 5\cdot 7} + C.$$

The value of $C$ is the area of the full quadrant of the curvilinear for each particular value of $m$; that is,

$$C = \int_0^{\frac{\pi}{2}} \sin^m x\cdot dx.$$

TABLE II

**Integrals of Even Powers of cos $x\cdot dx$, to Radius 1**

$$\int \cos^0 x\cdot dx = x.$$

$$\int \cos^2 x\cdot dx = \frac{x + \sin x\cdot\cos x}{2},$$

$$\int \cos^4 x\cdot dx = \frac{3x + 3\sin x\cdot\cos x + 2\sin x\cdot\cos^3 x}{2\cdot 4}$$

$$\int \cos^6 x\cdot dx = \frac{15x + 15\sin x\cdot\cos x + 10\sin x\cdot\cos^3 x + 8\sin x\cdot\cos^5 x}{2\cdot 4\cdot 6}.$$

$$\int \cos^8 x\cdot dx = (105x + 105\sin x\cdot\cos x + 70\sin x\cdot\cos^3 x + 56\sin x\cdot\cos^5 x + 48\sin x\cdot\cos^7 x) \div 2\cdot 4\cdot 6\cdot 8.$$

$$\int \cos^{10} x\cdot dx = (945x + 945\sin x\cdot\cos x + 630\sin x\cdot\cos^3 x + 504\sin x\cdot\cos^5 x + 432\sin x\cdot\cos^7 x + 384\sin x\cdot\cos^9 x) \div 2\cdot 4\cdot 6\cdot 8\cdot 10$$

**3.** This table may be extended by the following law of progression:

$$\int \cos^m x\cdot dx = \frac{m-1}{m}\int \cos^{m-2}x\cdot dx + \frac{1}{m}\sin x\cdot\cos^{m-1}x,$$

and for negative values of $m$, this expression may be changed to

$$\int \cos^{m-2}x\cdot dx = \frac{m}{m-1}\int \cos^m x\cdot dx - \frac{1}{m-1}\sin x\cdot\cos^{m-1}x,$$

and from this we get

$$\int \cos^{-2}x \cdot dx = \tan x.$$

$$\int \cos^{-4}x \cdot dx = \frac{2 \tan x + \dfrac{\tan x}{\cos^2 x}}{3}.$$

$$\int \cos^{-6}x \cdot dx = \frac{8 \tan x + \dfrac{4 \tan x}{\cos^2 x} + \dfrac{3 \tan x}{\cos^4 x}}{3 \cdot 5}.$$

$$\int \cos^{-8}x \cdot dx = \frac{48 \tan x + \dfrac{24 \tan x}{\cos^2 x} + \dfrac{18 \tan x}{\cos^4 x} + \dfrac{15 \tan x}{\cos^6 x}}{3 \cdot 5 \cdot 7}.$$

## Table III

### Integrals of Odd Negative Powers of Cos $x \cdot dx$, to Radius 1

$$\int \cos^{-1}x \cdot dx = \tfrac{1}{2} \log \frac{1 + \sin x}{1 - \sin x}.$$

$$\int \cos^{-3}x \cdot dx = \tfrac{1}{2} \log \frac{1 + \sin x}{1 - \sin x} + \frac{\tan x}{\cos x}.$$

$$\int \cos^{-5}x \cdot dx = \frac{\tfrac{3}{2} \log \dfrac{1 + \sin x}{1 - \sin x} + \dfrac{3 \tan x}{\cos x} + \dfrac{2 \tan x}{\cos^3 x}}{2 \cdot 4}.$$

$$\int \cos^{-7}x \cdot dx = \frac{\tfrac{15}{2} \log \dfrac{1 + \sin x}{1 - \sin x} + \dfrac{15 \tan x}{\cos x} + \dfrac{10 \tan x}{\cos^3 x} + \dfrac{8 \tan x}{\cos^5 x}}{2 \cdot 4 \cdot 6}.$$

The law of progression is

$$\int \frac{1}{\cos^{m+2}x} \cdot dx = \frac{m}{m-1} \int \frac{1}{\cos^m x} \cdot dx + \frac{1}{m+1} \cdot \frac{\tan x}{\cos^m x}$$

## Table IV

### Integrals of Odd Negative Powers of Sin $x \cdot dx$, to Radius 1.

$$\int \sin^{-1}x \cdot dx = -\tfrac{1}{2} \log \frac{1 + \cos x}{1 - \cos x} + C.$$

$$\int \sin^{-3}x \cdot dx = -\frac{\tfrac{1}{2} \log \dfrac{1 + \cos x}{1 - \cos x} + \dfrac{\cot x}{\sin x}}{2} + C.$$

$$\int \sin^{-5}x \cdot dx = -\frac{\tfrac{3}{2} \log \dfrac{1 + \cos x}{1 - \cos x} + \dfrac{3 \cot x}{\sin x} + \dfrac{2 \cot x}{\sin^3 x}}{2 \cdot 4} + C.$$

$$\int \sin^{-7}x \cdot dx = -\frac{\tfrac{15}{2} \log \dfrac{1 + \cos x}{1 - \cos x} + \dfrac{15 \cot x}{\sin x} + \dfrac{10 \cot x}{\sin^3 x} + \dfrac{8 \cot x}{\sin^5 x}}{2 \cdot 4 \cdot 6} + C.$$

The law of progression is

$$\int \frac{1}{\sin^{m+2}x}\cdot dx = \frac{m}{m+1}\int \frac{1}{\sin^m x}\cdot dx - \frac{1}{m+1}\cdot\frac{\cot x}{\sin^m x}.$$

The value of $C$ is the full quadrant of the curvilinear.

**4.** The value assigned to $C$ in the integrals of $\frac{1}{\sin^m x}\cdot dx$, in Tables I and IV, is readily derived from the following consideration. If we trace, for the full quadrant, the two curves, $y = \frac{1}{\sin^m x}$ and $y = \frac{1}{\cos^m x}$, having their origin at opposite ends of the axis of $x$, the two curves will coincide throughout. Therefore, the value of $\int \frac{1}{\sin^m x}\cdot dx$, given by Tables I and IV, simply measures negatively the value given by Tables II and III for $\int \frac{1}{\cos^m x}\cdot dx$. Hence, $\int \frac{1}{\sin^m x}\cdot dx$ measures negatively the complement of the area of the curvilinear; so that, adding to this negative quantity the full quadrant of the curvilinear gives the proper integral. These remarks equally apply to the curves $y = \tan^m x$ and $y = \frac{1}{\tan^m x}$, the integrals of which are given on another page.

*Series I*

The integral of $\sin^m x\cdot dx$, for any value of $m$, is $A\cdot(1 - \cos x) - \frac{B}{3}(1 - \cos^3 x) + \frac{C}{5}(1 - \cos^5 x) - \frac{D}{7}(1 - \cos^7 x) + \ldots$ where $A, B, C, D \ldots$ are the successive terms in the development of the binomial $(1+1)^{\frac{m-1}{2}}$; namely, $A = 1$; $B = \frac{m-1}{2}$;

$$C = \frac{m-1\cdot m-3}{2\cdot 4};\ D = \frac{m-1\cdot m-3\cdot m-5}{2\cdot 4\cdot 6}\ \ldots$$

**5.** Series I terminates with the term containing the $m$th power of $\cos x$, when $m$ is a positive odd integer; otherwise it is infinite. It therefore gives complete expressions for odd positive powers of $\sin x\cdot dx$.

*Series II*

The integral of $\cos^m x\cdot dx$, for any value of $m$, is $A\cdot\sin x - \frac{B}{3}\sin^3 x + \frac{C}{5}\sin^5 x - \frac{D}{7}\sin^7 x + \ldots$ where $A, B, C, D \ldots$ are the successive terms of the binomial $(1+1)^{\frac{m-1}{2}}$, as before.