ON A CATEGORY OF TRANSFORMATION GROUPS IN THREE AND FOUR DIMENSIONS

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ON A CATEGORY OF TRANSFORMATION GROUPS IN THREE AND FOUR DIMENSIONS.

INAUGURAL DISSERTATION

Submitted to the Philosophical Faculty of the University of Leipzig for the Degree of Doctor of Philosophy,

BY

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ITHACA, N. Y.
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TO MY MOTHER.

Written under the guidance of Professor Sophus Lie at the University of Leipzig.

Day of examination: July 26, 1898.

INTRODUCTION.

Riemann, in his well known publication on the "Hypothesen, welche der Geometrie zu Grunde liegen," started from his definition of an arc-element in a manifoldness of n dimensions,

$$ds = \sqrt{\Sigma a_{ik} dx_i dx_k},$$

in which the quantity under the radical is always positive and then proceeded to the development of the conception of constant measure of curvature, which is perhaps the substantial result of the publication. *Helmholtz* on the contrary goes still further back and attempts to found Riemann's assumption. He makes his own assumptions and, from these as a base, he attempts to prove that all the transformations possible under his conditions have the invariant

$$\Sigma a_{ik} dx_i dx_k.$$

Lie in an article in the "Leipziger Berichte," for October, 1886, notes inaccuracies in the development and points out that, under one interpretation of Helmholtz's axiom of unrestricted motion (freie Beweglichheit), his monodrom axiom is entirely superfluous, while under another interpretation, even all the axioms would be insufficient to determine the groups, which preserve the geometric qualities of a rigid body in space. Lie, in a later article in the same publication,[1] proves rigidly the

[1] *Leipziger Berichte*, Oct., 1890.

truth of his statement made in his article of 1886. He proceeds from Helmholtz's assumptions, with the exception of one, namely, the monodrom axiom and determines all the groups, satisfying the given conditions. Under the most general interpretation of the axiom of "unrestricted motion," he gets, besides the groups of Euclidian and non-Euclidian motion, five others. *Kowalewski*, a pupil of Lie, has in his inaugural dissertation[1] extended Lie's investigations in space of three dimensions to that of four and five. Besides the Euclidian and non-Euclidian groups, he finds three others that satisfy the condition of unrestricted motion. The discussion of these eight groups, or more particularly the one-parametric sub-groups, is the chief aim of this dissertation.

The behavior of points under certain special conditions is in some cases most remarkable. Sometimes all the points of a surface remain at rest and sometimes not, depending upon the choice of the points we hold stationary. In some cases too, we find all the points of a surface invariant wherever we may choose our points. Then, too, in space of four dimensions, we find one group has in some cases closed path-curves and in others not.

To give Lie's development in its entirety is of course out of the question, but at the same time, in order that we may have a proper insight into the groups, a short sketch is necessary.

[1] "Über eine Kategorie von Transformationsgruppen einer vierdimensionalen Mannigfaltigkeit." 1898.

CHAPTER I.

SKETCH OF LIE'S DEVELOPMENT OF THE GROUPS SATISFYING THE AXIOMS OF HELMHOLTZ.

Lie sums up the Helmholtz axioms in the following words :[1]

Let

$$x_1 = \rho\,(x, y, z, a_1 \; . \; . \; . \; . \; a_r)$$
$$y_1 = \phi\,(x, y, z, a_1 \; . \; . \; . \; . \; a_r)$$
$$z_1 = \psi\,(x, y, z, a_1 \; . \; . \; . \; . \; a_r)$$

be a set of real transformations in space of three dimensions under the following conditions :

(A) The functions ρ, ϕ, ψ are analytical functions of the variables and the parameters.

(B) Two points shall have one and only one invariant in the group.

(C) There shall be unrestricted motion in space, that is : the point $x_1\, y_1\, z_1$ can be transformed into every other point in space. If we keep $x_1\; y_1\; z_1$ fixed, then a second point can take ∞^2 positions. Hold two points stationary, then a third point can take ∞^1 positions. Finally, if we hold three points stationary, then all the points in space remain stationary.

(D) If we hold two points stationary and transform the remaining points in all possible ways, then the points, after traversing a finite distance, shall return at the same time to their initial positions.

[1] *Leipziger Berichte*, Oct., 1890.

That this set of transformations under the conditions A, B, C form a group with six parameters is evident. We have then the problem, to find all the six parametric groups in space, which are defined by real analytical equations and for all real points satisfy the conditions B and C.

The six equations:

$$W_k f = \xi_k (x_1, y_1, z_1) \frac{\delta f}{\delta x_1} \; . \; . \; . \; . \; . \; \zeta_k (x_2, y_2, z_2) \frac{\delta f}{\delta z_2} = 0$$

$$(k = 1, 2, \ldots 6).$$

can have one and only one solution. Therefore

$$\Delta_\pi = \Sigma \pm \xi_1 (x_1, v_1, z_1) \; . \; . \; . \; . \; \zeta_6 (x_2, y_2, z_2) = 0$$

while the 5 rowed sub-determinants do not all vanish.

We can put this criterion in another form. We multiply the $W_k f$ by such quantities $\psi_1 \; . \; . \; . \; . \; \psi_6$, that the co-efficients of $\frac{\delta f}{\delta x}, \frac{\delta f}{\delta y}, \frac{\delta f}{\delta z}$, in the expression $\psi_1 W_1 f + \; . \; . \; . \; . \; . \; + \psi_6 W_6 f$, shall be equal to zero. We obtain three equations in the ψ's with co-efficients depending on x_1, y_1, z_1, which we can solve for three of the ψ's. The other three are indeterminate. We choose three of the many expressions

$$V_i f = \Sigma \psi_{ik} W_k f$$

so that no relation

$$\omega_1 (x_1, y_1, z_1) V_1 f + \omega_2 V_2 f + \omega_3 V_3 f = 0$$

exists, and set

$$V_1 f = \xi_1^1 \frac{\delta f}{\delta x_2} + \eta_1^1 \frac{\delta f}{\delta y_2} + \zeta_1^1 \frac{\delta f}{\delta z_2}.$$