SOME INVARIANTS AND COVARIANTS OF TERNARY COLLINEATIONS. A DISSERTATION

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BY

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A DISSERTATION

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INTRODUCTION.

1. The analytical basis of the present paper is the form of Grassmann's Lückenausdruck which Gibbs called a dyadic. This, as the sequel shows, is merely a general bilinear function from which the variables are omitted. It may then represent a collineation or correlation and may be manipulated practically like the ordinary symbolical bilinear form.

Starting with this as a basis, the object is in the next place to give an interpretation by means of the invariant theory of various double products suggested by Gibbs and incidentally to obtain some of the properties of the invariants and covariants involved. The field of operation is plane projective geometry and the products are formed according to the combinatory multiplication of Grassmann.

Finally, in the third part, there is considered a skew symmetric function of any number of collineations which is called an alternant. It is a combinant, linear in the coefficients of each collineation, and presenting in some ways for functions of two sets of variables properties analogous to those of the expressions resulting from the combinatory multiplication of linear manifolds.

PART I. NOTATION.

I. *The open product or dyadic.*

2. In a space of two dimensions a sum of mixed products of similar construction, each containing a single factor x, may be written in the form

$$A_1x\cdot B_1 + A_2x\cdot B_2 + A_3x\cdot B_3,$$

where the dot is used to show that the order of multiplication is from left to right. A_i, B_i and x are geometric quantities, points or lines of the plane, and all products are formed according to the combinatory multiplication. This may be considered as resulting from the operation of x on the expression

$$A_1(\ )\cdot B_1 + A_2(\ )\cdot B_2 + A_3(\ )\cdot B_3,$$

the operation consisting in placing the variable x in the parentheses. This last expression is an example of what Grassman called an *open* product.*

* "Ausdehnungslehre" (1878), p. 265.

Gibbs wrote the open product in the form

$$A_1 B_1 + A_2 B_2 + A_3 B_3$$

and from the nature of its construction called it a *dyadic*.* The variable is supposed to operate on the dyadic from the outside and so give as result

$$xA_1 \cdot B_1 + xA_2 \cdot B_2 + xA_3 \cdot B_3$$

or

$$A_1 \cdot B_1 x + A_2 \cdot B_2 x + A_3 \cdot B_3 x$$

according as x is used as prefactor or postfactor.

In the present paper the notation of Gibbs will be used and combinatory products will be represented either by placing the letters in parentheses or by placing a bar over them. It is found convenient to use the parentheses when the product reduces to a scalar, or number, and in all other cases to use the bar. Unless otherwise expressly stated the variable will enter the dyadic as postfactor, i. e., the dyadic will operate on the variable. From analogy with the ordinary symbolism for a row product we shall write

$$AB = A_1 B_1 + A_2 B_2 + A_3 B_3.$$

It is to be observed that A_i and B_i in this expression have a definite size or intensity. If they are only projectively given the dyadic will have the form

$$AB = \lambda_1 A_1 B_1 + \lambda_2 A_2 B_2 + \lambda_3 A_3 B_3,$$

where the λ's are numbers determined when definite intensities are given to A_i and B_i.

3. As an operator the dyadic gives a linear transformation of quantities contragredient to B_i. For, x being such a quantity, since $(B_i x)$ is a number,

$$A(Bx) = \lambda_1 (B_1 x) A_1 + \lambda_2 (B_2 x) A_2 + \lambda_3 (B_3 x) A_3$$

which as a function of A_i is a simple manifold involving x linearly.

There are two cases of present interest. When A_i and B_i are contragredient we have a collineation; when cogredient, a correlation.

A dyadic of the form

$$a\alpha = \lambda_1 a_1 \alpha_1 + \lambda_2 a_2 \alpha_2 + \lambda_3 a_3 \alpha_3,$$

where the a's are points and the α's lines, represents a point collineation.† In

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* GIBBS's "Vector Analysis" (Wilson), chap. V.

† In the notation of CLEBSCH this is of course

$$(a\xi)(\alpha x) = \Sigma \lambda_i (a_i \xi)(\alpha_i x) = 0,$$

when a is given and ξ variable. If ξ were given and x variable the dyadic would be written αa. The dyadic is thus regarded not as giving a connex, but as setting up a definite transformation.

particular to the point $\overline{\alpha_2\alpha_3}$ corresponds the point

$$a(\alpha\alpha_2\alpha_3) = \lambda_1(\alpha_1\alpha_2\alpha_3)a_1.$$

To the vertices of the triangle α then correspond the points a_i. Since the triangle α merely presents a set of points to be operated upon it is obvious that this may be chosen at random, the collineation then determining a as its correspondent. While $a\alpha$ as an operator gives the collineation of points it involves internally the collineation of triads.

Similarly the dyadic

$$a\beta = \lambda_1 a_1\beta_1 + \lambda_2 a_2\beta_2 + \lambda_3 a_3\beta_3$$

represents a correlation in which the lines α_i correspond to the points of the triangle β. A like interpretation may be given for the dual cases αa and αb.

II. *Tetrad and counter-tetrad.*

4. We have seen that the dyadic in trinomial form involves the correspondence of triads. Since, however, a collineation or correlation in the plane is determined by four pairs of corresponding elements, it is of greater interest to have the dyadic involve a correspondence of sets of four. And as a dyadic always operates on a contragredient quantity this end can only be attained by the use of a self dual scheme of four-point and four-line.

With a 4-point we associate the 4-line obtained by taking the polar of each point with respect to the triangle of the other three. It is well known then that conversely the 4-point is obtained by taking the polar of each line with respect to the triangle of the other three. These mutually related systems have been called *tetrad* and *counter-tetrad*.*

Supposing the points a_i to satisfy only one linear relation (which is the only case of interest) their intensities may be chosen such that

(1) $$a_1 + a_2 + a_3 + a_4 = 0.$$

Operating on this identity with the products $\overline{a_i a_j}$ we find that the triple products $(a_i a_j a_k)$ are in absolute value all equal. If then we write

$$(a_2 a_3 a_4) = 4,$$

we obtain

(2) $$(a_i a_j a_k) = \pm 4 \qquad (i<j<k),$$

where the sign is positive or negative according as $a_i a_j a_k$ is complementary to an odd or an even term in the sequence $a_1 a_2 a_3 a_4$.

* F. Morley, *Trans. Amer. Math. Society*, vol. 4, p. 291.

Making use of these formulas the counter-tetrad α_i may be written in the canonical form

$$
(3)\qquad
\begin{aligned}
4\alpha_1 &= -\overline{a_2a_3} - \overline{a_3a_4} - \overline{a_4a_2},\\
4\alpha_2 &= \overline{a_4a_1} + \overline{a_1a_3} + \overline{a_3a_4},\\
4\alpha_3 &= -\overline{a_1a_2} - \overline{a_2a_4} - \overline{a_4a_1},\\
4\alpha_4 &= \overline{a_1a_2} + \overline{a_2a_3} + \overline{a_3a_1}.
\end{aligned}
$$

From these equations by addition we obtain

$$(4)\qquad \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0.$$

Multiplying the equations (3) by a_i and making use of (2) it is easily seen that

$$(5)\qquad (a_i\alpha_i) = 3, \qquad (a_i\alpha_j) = -1 \qquad (i \neq j).$$

We have here sixteen equations. From the identity

$$\sum_i (a_i\alpha_j) = \sum_j (a_i\alpha_j) = 0$$

it is seen however that seven of these equations are superfluous, their effect being to make a_i and α_i subject to conditions (1) and (4). When one tetrad is given there are then nine conditions to be satisfied by the other. And since a tetrad subject to the conditions (1) or (4) in addition to its eight geometrical constants involves an undetermined intensity it follows that there is a single solution. The equations (5) may therefore be taken as canonically defining a tetrad and counter-tetrad. Their symmetry in a_i and α_i indicates the mutuality previously mentioned.

From the equations (3) by direct multiplication we obtain

$$(6)\qquad \overline{\alpha_i\alpha_j} = a_l - a_k,$$

where i, j, k, l is a positive permutation of the numbers 1, 2, 3, 4. Multiplying by α_k and making use of (5)

$$(\alpha_i\alpha_j\alpha_k) = \pm 4 \qquad (i<j<k),$$

the sign being positive or negative according as $\alpha_i\alpha_j\alpha_k$ is an odd or even minor of $\alpha_1\alpha_2\alpha_3\alpha_4$. From the symmetry of the entire system in a_i and α_i we may finally write

$$(8)\qquad \overline{a_ia_j} = \alpha_l - \alpha_k,$$

where the rule of subscripts is the same as before.

5. The application of the preceding to the study of dyadics in the plane is now simple. Consider the collineation

$$(9)\qquad \sum (a_2a_3a_4)(\beta_2\beta_3\beta_4)a_1\beta_1.$$

Taking a_i and β_i subject to the conditions (2) and (7) the products $(a_i a_j a_k)(\beta_i \beta_j \beta_k)$ all become equal and (9) becomes

$$a_1\beta_1 + a_2\beta_2 + a_3\beta_3 + a_4\beta_4.$$

Operating on this with b_1, a point of the counter-tetrad of β, and making use of (5) we get

$$3a_1 - a_2 - a_3 - a_4 = 4a_1.$$

The points b_i pass by (9) into the points a_i. The collineation therefore transforms the associated system b, β into the system a, α and so the dyadic in this form involves a correspondence of tetrads.

In the same way we see that

(10) $$\sum (\alpha_2\alpha_3\alpha_4)(\beta_2\beta_3\beta_4)\alpha_1\beta_1$$

is a correlation which transforms the counter-tetrad of β into α and so carries the system b, β into the system α, a.

PART II. MULTIPLE PRODUCTS.

I. *Multiple products are complete invariants.*

6. With two dyadics AA', BB' is connected a form $\overline{AB}\,\overline{A'B'}$ which Gibbs called the double product of the two dyadics.* It is formed by multiplying the dyadics distributively, each pair of terms combining to form a product in which the prefactor is product of prefactors and postfactor product of postfactors. Gibbs showed that this double multiplication is distributive with respect to a resolution of either dyadic or is invariantive as is readily seen upon expansion.

So with a system of dyadics are a series of multiple products given by the various ways in which prefactors and postfactors may be independently combined. From their construction it is evident that such forms retain their significance when the prefactors and postfactors are transformed separately and therefore belong to the class of functions that Pasch called *complete* invariants.† If the dyadics appear as transformations operating on the elements of a certain field, since a transformation of postfactors amounts to a transformation of that field, it follows that the geometric interpretation of a multiple product must involve an arbitrary initial field. If, for example, a system of collineations and correlations in the plane operate upon four points, the multiple products will give results independent of the initial 4-point, i. e., invariants and covariants

* Loc. cit., p. 306.

The function here considered is a double product only in the sense that it is formed by a certain double process. It is neither the scalar nor the vector but the combinatorial double product. All of these have certain properties in common which characterize double multiplication.

† "Vollkommene Invariante," *Math. Ann.*, Bd. 52, p. 128.