

**A TREATISE ON  
ALGEBRAICAL  
GEOMETRY**

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A treatise on algebraical geometry by S. W. Waud

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**S. W. WAUD**

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A TREATISE

ON

ALGEBRAICAL GEOMETRY.

BY THE

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