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James K. Thurber

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A Geometric Algorithm for Solving the General Linear Programming Problem

§1. Introduction

The development of the subject of linear programming has centered to a large extent on the problem of maximizing (or minimizing) a linear form the variables of which are subject to linear inequalities or constraints. There have been proposed many procedures for carrying out such maximizations in an effective way, foremost of which is the simplex method of G. Dantzig. All of these methods have the objectionable feature that on occasion the number of steps required, though finite, becomes excessively large. Because of this (and in the absence of any alternative) the attitude assumed by many workers in the field is that a large collection of methods should be evolved in the hope that on all occasions at least one of them will prove practicable. In this spirit, a method is proposed in this paper which is essentially a "gradient" procedure, and yet has the added feature of concluding in a finite number of steps. In some problems it works more "efficiently" than the simplex method as a consequence of the fact that instead of moving between neighboring vertices of the polyhedron of "feasible solutions" (as in the simplex method) it provides for moving across faces of this polyhedron.

§2. The linear programming problem. Let:

E_n denote euclidean n-space;

A denote an mxn matrix with rows $R_1, \ldots, R_m$;

$b = (b_1, \ldots, b_m)$ be an m-dimensional (column) vector;

$p = (p_1, \ldots, p_n)$ be an n-dimensional (row) vector.

Then the general linear programming problem may be stated as follows:

Find $x \in E_n$ such that x maximizes

$$L(x) = p \cdot x = p_1x_1 + \ldots + p_nx_n ,$$

subject to the constraints

$$Ax \geq b .$$

(Note: The non-negativeness of the x_i's need not be among the constraints.)

§3. Further notation and its geometrical significance.

Let:

$|x|$ denote the length of a vector x E_n;

$D = \left\{ x | x \in E_n \text{ and } Ax \geq b \right\}$;

$D^* = \left\{ x | x \in D \text{ and } L(x) \geq L(u) \text{ for all } u \in D \right\}$.

The following is well known and easily verified.

Theorem 1. D and D^* are closed and convex.

If $x \in D^*$, then we call x a solution to our problem. If $x \in D$, then we call x a feasible solution.

Let $\tilde{A}$ be an rxn submatrix of A whose rows are $R_{i_1},\ldots,R_{i_r}$. Corresponding to this submatrix, let:

$\tilde{b}$ be the (column) vector $(b_{i_1},\ldots,b_{i_r})$;

$\tilde{F} = \{x \mid x \in D \text{ and } \tilde{A}x = \tilde{b}\}$;

$\tilde{V} = \{x \mid x \in E_n \text{ and } \tilde{A}x = 0\}$;

$\tilde{p}$ be the projection of p on V;

$$\tilde{q} = \begin{cases} p \cdot \tilde{p}/|\tilde{p}| & \text{if } \tilde{p} \neq 0 \\ 0 & \text{if } \tilde{p} = 0 \end{cases}.$$

(Note: We allow r = 0, in which case $\tilde{F} = D$, $\tilde{V} = E_n$, $\tilde{p} = p$, and $\tilde{q} = |p|$.)

Before launching our discussion of the algorithm, which will be presented in a completely algebraic context, it will be worthwhile to give geometric meaning to the symbols above, and to give a geometric description of the algorithm. For it is the geometry that motivates the method, and by interpreting our results geometrically at each stage of the discussion, the reader should have no trouble following along.

As it is well known, D is the intersection of halfspaces, and therefore a polyhedron (which may be bounded, unbounded or even empty). Then we may interpret $\tilde{F}$ as being a face of this polyhedron (which may be empty). Now suppose we have a point $x \in D$. For this x, L(x) has some value. Then what we would like to do, is to find a point $y \in D$ such that $L(y) > L(x)$. Furthermore, it would seem plausible, in moving from x to y, to do so in a manner that changes L(x) as much as possible for each unit of distance we move. Then of course, this direction