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QUATERNIONS AND PROJECTIVE GEOMETRY

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INTRODUCTION.

A QUATERNION q adequately represents a point Q to which a determinate weight is attributed, and, conversely, when the point and its weight are given, the quaternion is defined without ambiguity. This is evident from the identity

$$q = \left(1 + \frac{Vq}{Sq}\right)Sq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$

in which Sq is regarded as a weight placed at the extremity of the vector

$$\text{OQ} = \frac{Vq}{Sq} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

drawn from any assumed origin O. It is sometimes convenient to employ capitals Q concurrently with italics q to represent the same point, it being understood that

$$\text{Q} = \frac{q}{Sq} = 1 + \text{OQ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C).$$

Thus Q represents the point Q affected with a unit weight. The point O may be called the *scalar* point, for we have

$$\text{O} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D).$$

In order to develop the method, it becomes necessary to employ certain special symbols. With one exception these are found in Art. 365 of 'Hamilton's Elements of Quaternions,' though in quite a different connection. We write

$$(a, b) = bSa - aSb, \quad [a, b] = V \, . \, VaVb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E);$$

and in particular for points of unit weight, these become

$$(\text{A}, \text{B}) = \text{B} - \text{A}, \quad [\text{A}, \text{B}] = V.V\text{A}V\text{B} = V.V\text{A}\,.\,(\text{B} - \text{A}) \quad . \quad . \quad . \quad (F).$$

Thus (ab) is the product of the weights $SaSb$ into the vector connecting the points, and $[ab]$ is the product of the weights into the moment of the vector connecting the points with respect to the scalar point. The two functions (ab) and $[ab]$ completely define the line ab.

Again HAMILTON writes

$$[a,b,c]=(a,b,c)-[b,c]\mathrm{S}a-[c,a]\mathrm{S}b-[a,b]\mathrm{S}c;\ (a,b,c)=\mathrm{S}[a,b,c]=\mathrm{SV}a\mathrm{V}b\mathrm{V}c. \quad \text{(G)};$$

or if we replace a, b, c by $(1+\alpha)\mathrm{S}a$, $(1+\beta)\mathrm{S}b$, $(1+\gamma)\mathrm{S}c$, where α, β and γ are the vectors from the scalar point to three points a, b and c, we have

$$[\mathrm{A}, \mathrm{B}, \mathrm{C}] = \mathrm{S}\alpha\beta\gamma - \mathrm{V}(\beta\gamma+\gamma\alpha+\alpha\beta);\ (\mathrm{A}, \mathrm{B}, \mathrm{C}) = \mathrm{S}\alpha\beta\gamma \quad . \quad . \quad . \quad \text{(H)}.$$

Hence it appears that $[a, b, c]$ is the symbol of the plane a, b, c; for $-\mathrm{V}[a, b, c](a, b, c)^{-1}$ is the reciprocal of the vector perpendicular from the scalar point on that plane. Also $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ is the sextupled volume of the tetrahedron OABC.

Again, HAMILTON writes for four quaternions

$$(abcd) = \mathrm{S}\,.\,a[bcd] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)};$$

and in terms of the vectors this is seen to be the products of the weights into the sextupled volume of the pyramid (ABCD).

Other notations may of course be employed for these five combinatorial functions of two, three, or four quaternions or points, but HAMILTON'S use of the brackets seems to be quite satisfactory.

In the same article HAMILTON gives two most useful identities connecting any five quaternions. These are

$$a(bcde) + b(cdea) + c(deab) + d(eabc) + e(abcd) = 0\,. \quad . \quad . \quad . \quad \text{(J)},$$

and

$$e(abcd) = [bcd]\mathrm{S}ae - [acd]\mathrm{S}be + [abd]\mathrm{S}ce - [abc]\mathrm{S}de \quad . \quad . \quad . \quad \text{(K)},$$

which enable us to express any point in terms of any four given points, or in terms of any four given planes.

The equation of a plane may be written in the form

$$\mathrm{S}lq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(L)};$$

and thus l, any quaternion whatever, may be regarded as the symbol of a plane as well as of a point.

On the whole, it seems most convenient to take as the auxiliary quadric the sphere of unit radius

$$\mathrm{S}\,.\,q^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(M)},$$

whose centre is the scalar point. With this convention the plane $\mathrm{S}lq = 0$ is the polar of the point l with respect to the auxiliary quadric; or the plane is the reciprocal of the point l. Thus the principle of duality occupies a prominent position.

The formulæ of reciprocation

$$([abc];[abd]) = [ab](abcd);\,[[abc];[abd]] = -(ab)(abcd)\,. \quad . \quad . \quad \text{(N)}$$

connecting any four quaternions are worthy of notice, and are easily proved by replacing the quaternions by $1 + \alpha$, $1 + \beta$, $1 + \gamma$, and $1 + \delta$ respectively. In complicated relations it may be safer to separate the quaternions as in these formulæ by semi-colons, but generally the commas or semi-colons may be omitted without causing any ambiguity.

These new interpretations are not in the least inconsistent with any principle of the calculus of quaternions. We are still at liberty to regard a quaternion as the separable sum of a vector and a scalar, or as the ratio or product of two vectors, or as an operator, as well as a symbol of a point or of a plane.

In particular, in addition to HAMILTON's definition of a vector as a right line of given direction and of given magnitude, and in addition to his subsequent interpretations of a vector as the ratio or product of two mutually rectangular vectors, or as a versor, we may now consider a vector as denoting the point at infinity in its direction, or the plane through the centre of reciprocation. For the vector OQ of equation (B) becomes infinitely long if $Sq = 0$, and the plane $Slq = 0$ passes through the scalar point if $Sl = 0$. We may also observe that the difference of two unit points A − B is the vector from one point B to the other A, and this again is in agreement with the opening sections of the "Lectures."

Additional illustrations and examples may be found in a paper on "The Interpretation of a Quaternion as a Point-symbol," 'Trans. Roy. Irish Acad.,' vol. 32, pp. 1–16.

The only other symbols peculiar to this method are the symbols for quaternion arrays. The five functions (ab), $[ab]$, $[abc]$, (abc), and $(abcd)$ are particular cases of arrays, being, in fact, arrays of one row. In general the array of m rows and n columns

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & \ldots & a_n \\ b_1 & b_2 & b_3 & \ldots & b_n \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ p_1 & p_2 & p_3 & \ldots & p_n \end{matrix}\right\} \quad \ldots \ldots \ldots \quad (O)$$

may be defined as a function of mn quaternion constituents, which vanishes if, and only if, the groups of the constituents composing the rows were connected by linear relations with the same set of scalar multipliers. In other words, the array vanishes if scalars $t_1, t_2 \ldots t_n$ can be found to satisfy the m equations

$$\begin{aligned} t_1 a_1 + t_2 a_2 + \ldots + t_n a_n &= 0, \\ t_1 b_1 + t_2 b_2 + \ldots + t_n b_n &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \ldots \quad \cdot \quad & \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \ldots \quad \cdot \quad & \cdot \quad \cdot \\ t_1 p_1 + t_2 p_2 + \ldots + t_n p_n &= 0. \end{aligned}$$

The expansion of arrays is considered in a paper on "Quaternion Arrays," 'Trans. Roy. Irish Acad.,' vol. 32, pp. 17–30.

SECTION I.

FUNDAMENTAL GEOMETRICAL PROPERTIES OF A LINEAR QUATERNION FUNCTION.

1. The quaternion equation

$$f(p+q) = fp + fq \qquad (1),$$

may be regarded as a definition of the nature of a linear quaternion function f, the quaternions p and q being perfectly arbitrary. As a corollary, if x is any scalar,

$$f(xp) = xfp \qquad (2),$$

and on resolving fq in terms of any four arbitrary quaternions a_1, a_2, a_3, a_4, we must have an expression of the form

$$fq = a_1 Sb_1 q + a_2 Sb_2 q + a_3 Sb_3 q + a_4 Sb_4 q \qquad (3),$$

because the coefficients of the four quaternions a must be scalar and distributive functions of q. Sixteen constants enter into the composition of the function f, being four for each of the quaternions b.

2. When a quaternion is regarded as the symbol of a point, the operation of the function f produces a linear transformation of the most general kind.

The equations

$$f(xa + yb) = xfa + yfb; \quad f(xa + yb + zc) = xfa + yfb + zfc \qquad (4),$$

show that the right line a, b is converted into the right line fa, fb, and the plane containing three points a, b, c into the plane containing their correspondents, fa, fb and fc.

The homographic character of the transformation is also clearly exhibited.

3. In order to specify a function of this kind it is necessary to know the quaternions a', b', c', d' into which any set of four unconnected quaternions, a, b, c, d, are converted. Thus, from the identical relation

$$q(abcd) + a(bcdq) + b(cdqa) + c(dqab) + d(qabc) = 0 \quad . \quad . \quad . \quad (5),$$

connecting one arbitrary quaternion with the four given quaternions, is deduced the equation

$$fq(abcd) + a'(bcdq) + b'(cdqa) + c'(dqab) + d'(qabc) = 0 \quad . \quad . \quad . \quad (6),$$

which determines the result of operating by f on q.

When we are merely concerned with the geometrical transformation of points, the absolute magnitudes* of the representative quaternions cease to be of importance, and the function

$$fq = x\text{A}'(\text{BCD}q) + y\text{B}'(\text{CD}q\text{A}) + z\,\text{C}'(\text{D}q\text{AB}) + w\text{D}'(q\text{ABC}) \quad . \quad . \quad . \quad . \quad (7),$$

which involves four arbitrary scalars, converts the four points A, B, C, D into four others, A′, B′, C′, ′D. Given a fifth point E and its correspondent E′, the four scalars are determinate to a common factor, and subject to a scalar multiplier, the function which produces the transformation is

$$fq = \text{A}'(\text{BCD}q)\,.\,\frac{(\text{B}'\text{C}'\text{D}'\text{E}')}{(\text{BCDE})} + \text{B}'(\text{CD}q\text{A})\,.\,\frac{(\text{C}'\text{D}'\text{E}'\text{A}')}{(\text{CDEA})} + \text{C}'(\text{D}q\text{AB})\,.\,\frac{(\text{D}'\text{E}'\text{A}'\text{B}')}{(\text{DEAB})}$$
$$+ \text{D}'(q\text{ABC})\,.\,\frac{(\text{E}'\text{A}'\text{B}'\text{C}')}{(\text{EABC})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

It is only necessary to replace q by E in order to verify this result.

4. *A linear quaternion function, f, being regarded as effecting a transformation of points, the inverse of its conjugate f'^{-1} produces the corresponding tangential transformation.*

For any two quaternions, p and q,

$$\text{S}pq = \text{S}pf^{-1}q' = \text{S}f'^{-1}pq' = \text{S}p_{\prime}q' \text{ if } q' = fq,\ p_{\prime} = f'^{-1}p \quad . \quad . \quad . \quad (9).$$

Hence any plane $\text{S}pq = 0$, in which the quaternion q represents the current point, transforms into the plane $\text{S}p_{\prime}q' = 0$, and the proposition is proved.

Thus, when symbols of points (q) are transformed by the operation of f, symbols of planes (p), or of points reciprocal to the planes, are transformed by the operation of f'^{-1}.

5. HAMILTON's beautiful method of inversion of a linear quaternion function receives a geometrical interpretation from the results of the last article.

* In accordance with the notation proposed ('Trans. Roy. Irish Acad.,' vol. 32, p. 2), capital letters are used in this article concurrently with small letters to denote the same points, but the weights for the capital symbols are unity; thus $q = \text{Q}\text{S}q = (1 + o\text{Q})\,\text{S}q$.