COMPANION TO ALGEBRA: WITH NUMEROUS EXAMPLES

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649488414

Companion to Algebra: With Numerous Examples by L. Marshall

Edited by Trieste Publishing Pty Ltd.
Cover @ 2017

www.triestepublishing.com

L. MARSHALL

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Trieste

Companion to Algebra

WITH NUMEROUS EXAMPLES

BY

L. MARSHALL, M.A.

ASSISTANT MASTER AT CHARTERHOUSE

RIVINGTONS

WATERLOO PLACE, LONDON

MDCCCLXXXIII

PREFACE.

THIS work is intended especially for those candidates preparing for the Woolwich Entrance and similar examinations, who are already familiar with the easier parts of Algebra, as given in any of the elementary text-books, but are not equal to attacking the larger and more systematic treatises. I have endeavoured to make the selection of Theorems and Examples as interesting and useful as possible, and I have ventured, on account of the importance of the results, to give one or two proofs which are not as logically complete as might be wished. Those of the examples which are not original I have obtained from Army Entrance, University and School Examination Papers.

L. MARSHALL.

CHARTERHOUSE, GODALMING, 1882.

COMPANION TO ALGEBRA.

I.—ELEMENTARY FORMULÆ AND RESULTS.

The following formulæ in §§ 1 and 3-9 can be directly verified by multiplication, and should be committed to memory.

1. $(a \pm b)^2 = a^2 \pm 2ab + b^2$.

e.g. $(2ax - 3y^2)^2 = (2ax)^2 - 2 \cdot 2ax \cdot 3y^2 + (3y^2)^2 = 4a^2x^2 - 12axy^2 + 9y^4$.

Notice that $(a-b)^2 = (b-a)^2$.

2. $(a+b+c+d+\ldots+k)^2 =$ *sum of squares of* $a, b, \ldots k$ + *twice the product of each pair, formed by taking each term in turn with each of those that follow it.*

Proof. $(a+b+c+d+\ \ldots\ +k)^2 = (a+\overline{b+c+\ \ldots\ +k})^2$

$$= a^2 + 2a(b+c+\ \ldots\ +k) + (b+\overline{c+\ \ldots\ +k})^2$$

$$= a^2 + 2a(b+\ \ldots\ +k) + b^2 + 2b(c+\ \ldots\ +k) + (c+\overline{d+\ \ldots\ k})^2,$$

and so on till we get the squares of all the letters.

It will be seen that this result may also be arranged in the following way:—

$$(a+b+c+d+\ \ldots\ +k)^2 = a^2 + (2a+b)b + (2a+2b+c)c + \ldots$$
$$\ldots + (2a+2b+2c+\ \ldots\ +k)k.$$

$$e.g. \left(a - 2b + \frac{c^3}{2} - d^2\right)^2 = a^2 + 4b^2 + \frac{c^6}{4} + d^4 + 2a(-2b) + 2a \cdot \frac{c^3}{2} + 2a(-d^2)$$
$$- 4b \cdot \frac{c^3}{2} - 4b(-d^2) + c^3(-d^2)$$
$$= a^2 + 4b^2 + \frac{c^6}{4} + d^4 - 4ab + ac^3 - 2ad^2 - 2bc^3 + 4bd^2 - c^3d^2.$$

3. $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$.

4. $(a \pm b)^4 = a^4 \pm 4a^3b + 6a^2b^2 \pm 4ab^3 + b^4$.

5. $(a+b)(a-b) = a^2 - b^2$, *i.e. the product of the sum and difference of two quantities = the difference of their squares.*

$$e.g.\ (127)^2 - (123)^2 = (127+123)(127-123) = 250 \times 4 = 1000.$$
$$9c^2 - 4(a-b)^2 = (3c+2a-2b)(3c-2a+2b).$$
$$(a-b+c+d)(a+b-c+d) = \{(a+d)-(b-c)\}\{(a+d)+(b-c)\}$$
$$= (a+d)^2 - (b-c)^2 = a^2 + 2ad + d^2 - b^2 + 2bc - c^2.$$

6. $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

$$e.g.\ 8a^3 + \frac{b^6}{27} = (2a)^3 + \left(\frac{b^2}{3}\right)^3 = \left(2a + \frac{b^2}{3}\right)\left(4a^2 - \frac{2ab^2}{3} + \frac{b^4}{9}\right).$$

7. $a^4 + a^2b^2 + b^4 = (a^2 + ab + b^2)(a^2 - ab + b^2)$.

$$e.g.\ x^8 + x^4 + 1 = (x^4 + x^2 + 1)\ (x^4 - x^2 + 1)$$
$$= (x^2 + x + 1)\ (x^2 - x + 1)(x^4 - x^2 + 1).$$

8. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab)$.

9. $(x+a)(x+b) = x^2 + (a+b)x + ab$.

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$$
$$(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3$$
$$+ (ab+ac+ad+bc+bd+cd)x^2 + (bcd+acd+abd+abc)x + abcd.$$

Notice that the co-efficient of x in the last formula is obtained by leaving out each letter in turn.

$$e.g.\ \left(x + \frac{1}{2}\right)\left(x + \frac{1}{3}\right) = x^2 + \frac{5}{6}x + \frac{1}{6}\ ;\ (x^2y^2 - xy - 12) = (xy-4)(xy+3)\ ;$$
$$(x+1)(x+2)(x-3) = x^3 + (1+2-3)x^2 + (2-3-6)x - 6 = x^3 - 7x - 6.$$

10. $\dfrac{x^n + y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + y^{n-1}$, when n is *odd.*

$\dfrac{x^n - y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots - y^{n-1}$, when n is *even.*

$\dfrac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}$, when n is either *odd* or *even.*

The last three formulæ may be stated otherwise, and proved by *induction* thus :—

e.g. $x^n + y^n$ is divisible by $x+y$ when n is odd.

For $\dfrac{x^n+y^n}{x+y}=x^{n-1}-x^{n-2}y+\dfrac{x^{n-2}+y^{n-2}}{x+y}\cdot y^2$ by actual division; therefore, if $x^{n-2}+y^{n-2}$ is divisible by $x+y$, so also is x^n+y^n, but we know (§ 6) that x^3+y^3 is divisible by $x+y$, therefore x^5+y^5 is also, and so by induction is x^n+y^n when n is any *odd* integer.

These results are most easily remembered by the simple cases of §§ 5, 6.

11. The following important results are proved on the supposition that the quantities a, b, c, d are all positive, and that $a>b$, $b>c$, $c>d$.

(A.) To prove (i.) $a+(b-c)=a+b-c$ and (ii.) $a-(b-c)=a-b+c$

(i.) To get $a+(b-c)$, we have to add $(b-c)$ to a. Now, if we add b to a, we get $a+b$, but we should have added c too much. Therefore for the true result we must subtract c; we thus get $a+(b-c)=a+b-c$.

(ii.) To get $a-(b-c)$, we have to subtract $(b-c)$ from a. Now, if we subtract b from a, we get $a-b$, but we should have subtracted c too much. Therefore we must add c to the result: we thus get $a-(b-c)=a-b+c$.

Similarly, $a+(b+c)=a+b+c$ and $a-(b+c)=a-b-c$

(B.) To prove the *Law of Signs* in multiplication, *i.e.* that

$$(a-b)(c-d)=ac-bc-ad+bd.$$

Put $a-b=M$, then M is a *positive* quantity, $\because a>b$ by supposition.

$$\therefore (a-b)(c-d)=M(c-d).$$

Now, to get M multiplied by $(c-d)$, if we take Mc, that is M multiplied by c, we are multiplying by c instead of $c-d$, that is by d times too much; therefore, to get the true result, we must subtract M multiplied by d, *i.e.* dM.

$$\text{Therefore } M(c-d)=cM-dM$$

$$\therefore (a-b)(c-d)=c(a-b)-d(a-b)$$

$$=(ac-bc)-(ad-bd) \text{ as before}$$

$$=ac-bc-ad+bd \text{ by (A).}$$