1. THE SYZYGETIC PENCIL OF CUBICS WITH A NEW GEOMETRICAL DEVELOPMENT OF ITS HESSE GROUP, G216; 2. THE COMPLETE PAPPUS HEXAGON: DISSERTATION

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CHARLES CLAYTON GROVE

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I. The Syzygetic Pencil of Cubics with a new Geometrical Development of its Hesse Group, G_{216}.

II. The Complete Pappus Hexagon.

DISSERTATION

SUBMITTED TO THE BOARD OF UNIVERSITY STUDIES OF THE JOHNS HOPKINS UNIVERSITY IN CONFORMITY WITH THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

BY
CHARLES CLAYTON GROVE

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INTRODUCTION.

1. The course of lectures by Prof. Frank Morley during the winter of 1903–4 on cubic curves suggested this dissertation and prepared me to carry on the research. The trend was largely determined by an incidental question by Prof. A. Cohen as to the groups involved in the system of conics which I had just presented to the Mathematical Seminary. The interest and valuable sug. gestions of Dr. A. B. Coble in the carrying on of the work are gratefully acknowledged.

2. The close connection between the Hesse group and the syzygetic pencil of cubics makes it necessary to say at least something about this pencil of curves. Without attempting even an outline of the theory, I present in Section I. only such matter as is needed later, besides some new facts concerning the pencil and a figure showing the appearance of some noteworthy and specially related cubics of the pencil. No figure seems ever to have been published except that in connection with the paper of Prof. Morley in the Proceedings of the London Math-Society, Ser. 2, Vol. 2, Part 2, which shows arbitrarily selected cubics. The initial and all but the closing work leading to that figure was done by me. Therefore, I present a figure of the pencil herein, also one of the corresponding polar-reciprocal range of line cubics.

Section II. shows how to derive a closed system of thirty-six conics analogous to the conic of Section I. as to which the pencil and range are polar-reciprocal. It also discusses the action of the polarities of these conics upon the four inflexional triangles, and presents some history of similar considerations.

In Section III. there is given a brief history of the attempts to determine all finite groups of transformations, and in particular an account of the Hesse Group of 216 collineations. Further, we derive and write down the matrices of these collineations by means of the closed system of thirty-six conics, which are here differently defined than in Section II. and are given accordingly. All the sub-groups are found and discussed. The collineations are finally classified as to periodicity.

Section IV. treats of triangles in other perspective forms than six-fold as are the inflectional triangles above. As a second way to secure triangles in three-fold perspective, also some in two- and one-fold perspective, we develop what we call the Complete Pappus Hexagon, in its dualistic forms and deduce a number of theorems connected with it.

I.

THE SYZYGETIC PENCIL. DRAWINGS OF THE PENCIL AND RANGE.

1. The name SYZYGETIC[1] is given to the pencil of cubics determined by a cubic f and its Hessian Δ,

$$\varkappa f + \lambda \Delta = 0, \tag{1}$$

which has the same nine inflexions for all its cubics, the intersections of the two cubics f and Δ, and so has the same four inflexional triangles.

It is well known that any non-singular cubic may be brought in four ways, as shown for example by Weber,[2] into Hesse's canonical form

$$x_1^3 + x_2^3 + x_3^3 + 6\,m x_1 x_2 x_3 = 0. \tag{2}$$

This simply means that the cubic has been referred to one of its inflexional triangles as reference triangle. The Hessian covariant of the form (2) is

$$\Delta \equiv -\,m^2 (x_1^3 + x_2^3 + x_3^3) + (1 + 2\,m^3)\, x_1 x_2 x_3 . \tag{3}$$

Its vanishing gives the Hesse Cubic or Hessian. Thus if m is the parameter of the cubic (2) and m' is that of its Hessian, we have $6\,m' = -\dfrac{1 + 2\,m^3}{m^2}$; and form (2) for all values of m from $-\infty$ to $+\infty$ gives the pencil as well as form (1).

The relation between m and m' shows that *each cubic of the pencil has but one Hessian but is Hessian to three cubics of the pencil.*

Choosing one of the inflexional triangles as reference triangle, we readily calculate in turn,

The coördinates of the nine inflexions,

Their arrangement on the sides of each of four triangles,[3]

The equations of the sides and opposite vertices of these triangles,

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[1] Clebsch-Lindemann: Leçons sur la Géométrie, II, p. 230.

[2] Lehrbuch der Algebra, 2. Aufl., II, §§ 106, 107; ss. 399–404.

[3] C.-L.: II, (7) p. 233.

The polar conics of the inflexions, which are in each case[1] two right lines, viz., the inflexional tangent and the harmonic polar of the inflexion.

For future reference we give the sides of the inflexional triangles:

A.	B.	C.	D.	
$x_1 = 0.$	$x_1 + x_2 + x_3 = 0.$	$\omega^2 x_1 + x_2 + x_3 = 0.$	$\omega x_1 + x_2 + x_3 = 0.$	
$x_2 = 0.$	$x_1 + \omega^2 x_2 + \omega x_3 = 0.$	$x_1 + \omega^2 x_2 + x_3 = 0.$	$x_1 + \omega x_2 + x_3 = 0.$	(4)
$x_3 = 0.$	$x_1 + \omega x_2 + \omega^2 x_3 = 0.$	$x_1 + x_2 + \omega^2 x_3 = 0.$	$x_1 + x_2 + \omega x_3 = 0.$	

The equations of the vertices respectively *opposite* are given by exchanging ξ for x and interchanging ω and ω^2. ω and ω^2 are the complex cube roots of unity.

2. Some Particular Cubics of the Pencil.

(*a*) The cubics whose parameter m is respectively $\infty, -\frac{1}{2}, -\frac{1}{2}\omega^2, -\frac{1}{2}\omega$ are the four inflexional triangles in the order as obtained above, of which[2] two are real and two imaginary. The polar line[3] as to these triangles of any point y has coordinates respectively,

	x_1	x_2	x_3
A:	$y_2 y_3$	$y_3 y_1$	$y_1 y_2.$
B:	$y_1^2 - y_2 y_3$	$y_2^2 - y_3 y_1$	$y_3^2 - y_1 y_2.$
C:	$y_1^2 - \omega^2 y_2 y_3$	$y_2^2 - \omega^2 y_3 y_1$	$y_3^2 - \omega^2 y_1 y_2.$
D:	$y_1^2 - \omega y_2 y_3$	$y_2^2 - \omega y_3 y_1$	$y_3^2 - \omega y_1 y_2.$

The determinant formed from any three rows of these coefficients vanishes identically, therefore the four polars pass through a common point, so we say,

The four polar lines of any point as to the four inflexional triangles meet in a point, or also

Any two of the inflexional triangles are apolar as seen from any point of the plane, and are thus syzygetic.

(*b*) The first of two covariant cubics of the pencil is the Hessian, equation (3), p. 5. The second is the Cayleyan of the cubic (2). The polar conic as to this cubic of a point y is

$$y_1(x_1^2 + 2m x_2 x_3) + y_2(x_2^2 + 2m x_3 x_1) + y_3(x_3^2 + 2m x_1 x_2) = 0.$$

Considering the y's as parameters, this is a *net of polar conics*. By inspection, we see that the conic $m\xi_1^2 - \xi_2\xi_3$ is apolar with the three conics of the net;

[1] C.-L.: II, p. 227; Salmon: Higher Plane Curves, 3. Ed., §§ 74, 170, pp. 59, 146.
[2] C.-L.: II, p. 230, also pp. 289, 310.
[3] H. P. C., § 165, p. 143.