MATHEMATICAL TRACTS FOR PHYSICISTS: INTRODUCTION TO THE CALCULUS OF VARIATIONS

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MATHEMATICAL TRACTS FOR PHYSICISTS

INTRODUCTION
TO THE
CALCULUS OF VARIATIONS

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CHAPTER I

INTRODUCTION

1. The Calculus of Variations owed its origin to the attempt to solve a very interesting and rather narrow class of problems in Maxima and Minima, in which it is required to find the form of a function such that the definite integral of an expression involving that function and its derivative shall be a maximum or a minimum.

Let us consider three simple examples: The Shortest Line, The Curve of Quickest Descent, and The Minimum Surface of Revolution.

(*a*) *The Shortest Line.* Let it be required to find the equation of the shortest plane curve joining two given points.

We shall use rectangular coördinates in the plane in question taking one of the points as the origin. Call the coördinates of the second point x_1, y_1.

If $y = f(x)$ is a curve through $(0, 0)$ and (x_1, y_1) and I is the length of the arc between the points, obviously

$$I = \int_0^{x_1} \sqrt{dx^2 + dy^2}$$

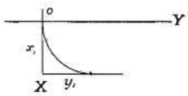
or

$$I = \int_0^{x_1} \sqrt{1 + y'^2}dx, \tag{1}$$

and we wish to determine the form of the function f so that this integral shall be a minimum.

(*b*) *The Curve of Quickest Descent.* Let it be required to find the form of a smooth curve lying in a vertical plane and joining two given points, down which a particle starting from rest will slide under gravity from the first point to the second in the least possible time.

We shall use rectangular axes in the vertical plane taking the higher point as the origin and taking the axis of X downward. Call the coördinates of the second point x_1, y_1.



Let $y = f(x)$ be a curve through $(0, 0)$ and (x_1, y_1) and use the well-known fact that $\frac{ds}{dt}$, the velocity of the moving particle at any time, is $\sqrt{2gx}$.

We have $$\frac{ds}{dt} = \sqrt{2gx},$$

whence $$dt = \frac{ds}{\sqrt{2gx}} = \frac{\sqrt{1 + y'^2}}{\sqrt{2gx}}\,dx$$

and $$t = \int_0^{x_1} \frac{1}{\sqrt{2gx}} \sqrt{1 + y'^2}dx.$$

Let $$I = \int_0^{x_1} \frac{1}{\sqrt{x}} \sqrt{1 + y'^2}dx, \tag{2}$$

and the form of the function f is to be determined so that this integral shall be a minimum.

(*c*) *The Minimum Surface of Revolution.* Given two points and a line which are co-planar, let it be required to find the form of a curve terminated by the two points and lying in the plane which, by its revolution about the given line, shall generate a surface of the least possible area.

Take the line as the axis of X and use an axis of Y through one of the points. Call the coördinates of the points 0, y_0, and x_1, y_1. Let $y = f(x)$ be a curve through $(0, y_0)$ and (x_1, y_1).

If S is the area of the surface of revolution generated by the curve,

$$S = 2\pi \int_0^{x_1} y ds = 2\pi \int_0^{x_1} y \sqrt{1 + y'^2} dx.$$

Let
$$I = \int_0^{x_1} y \sqrt{1 + y'^2} dx, \tag{3}$$

and we wish to determine the form of the function f so that I shall be a minimum.

2. The three problems just considered are special cases of what we shall call our *fundamental* problem which is, to determine the form of the function f so that if $y = f(x)$, $\int_{x_0}^{x_1} \phi \, (x, y, y') \, dx$ shall be a maximum or a minimum; ϕ being a given function and x_0 and x_1 being given constants, as are y_0 and y_1, the corresponding values of y.

In ordinary problems in maxima and minima $y = f(x)$ is a given function and we wish to find a value, x_0, of x for which y is greater, if we seek a maximum, less, if we seek a minimum, than for *neighboring* values of x; that is, for values of x differing from x_0 by a sufficiently small amount whether that amount is positive or negative.

In our new problems, to speak in geometrical language, we have to find the *form* of a curve for which our integral, I, is greater or less than for any *neighboring* curve having the same end-points.

3. Let us now attack our first problem, that of the *shortest line*. We have to find the form of the function f so that if

$$I = \int_0^{x_1} \sqrt{1 + y'^2} \, dx$$

I shall be a minimum when $y = f(x)$. *v.* Art. 1 (*a*).