A RUDIMENTARY TREATISE ON THE INTEGRAL CALCULUS

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RUDIMENTARY TREATISE
ON THE
INTEGRAL CALCULUS.

BY HOMERSHAM COX, B.A.
BARRISTER-AT-LAW;
AUTHOR OF A "MANUAL OF THE DIFFERENTIAL CALCULUS."

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PREFACE.

The Differential and the Integral Calculus have been established upon entirely different axioms and definitions by the several founders of those sciences. The primary ideas of infinitesimals, fluxions, and exhaustions, though their results coincide, for the simple reason that all pure truth is consistent with itself, are widely diverse in their abstract nature. In writing, therefore, on the principles of either Calculus, a difficulty presents itself in the necessity of electing between systems, each of which has the sanction of high authority and peculiar intrinsic merits.

This consideration is of especial importance in a "Rudimentary Treatise," which cannot, of course, fulfil the profession of its title without singleness and simplicity of its fundamental ideas, and an exactness of thought and language often very difficult of attainment. The choice of methods in the present work has been determined partly by historical considerations. The discoverers of new truths usually search after them by the simplest and most familiar considerations; and it seems natural to presume that, as far at least as abstract principles are concerned, the way of discovery is the easiest way of instruction.

The original idea upon which Newton based the system of fluxions, regarded a differential coefficient as the rate of increase of a function. The idea upon which Leibnitz and the Bernouillis established the Integral Calculus, regarded an integral as the limit of the summation of an indefinite number of indefinitely diminishing quantities. The facility with which the idea of "rate" may be conceived and applied to the science of which Newton was the great founder, and the similar advantages of the idea of summation in the Integral Calculus, determined the selection of the first idea as the basis of the "Manual of the Differential Calculus" by the present writer, and the second as the basis of the present treatise.

The value and importance of what is termed by Professor De Morgan the "summatory" definition of integration, has been insisted upon by him and others of the most eminent modern mathematicians; but the present is probably an almost solitary attempt to establish the Integral Calculus on that definition exclusively. Throughout the entire range of the practical applications of the Integral Calculus—to Geometry, Mechanics, &c.—the idea of summation is solely and universally applied. The rival definition of the Integral Calculus—as the inverse of the Differential Calculus—has a merely relative signification, and is, therefore, essential only in analytical investigations of the relations of the two sciences.

But whatever system be adopted for establishing either calculus must of necessity involve the idea of limits and limiting values. An unreasonable reluctance has been sometimes exhibited in adopting this idea in elementary treatises, whereas that it is one by no means difficult to be conceived is shewn by its adoption in the first ages of mathematics. By far greater difficulties have arisen from the shifts to which resort has been had to evade it in theorems of which demonstrations without it are necessarily illogical.

The idea of limits occurs, or ought to occur, much earlier in the study of exact science than is generally allowed. This idea is essentially involved in Arithmetic, Euclid, and Algebra. The laws of operation with recurring decimals and surds cannot be accurately established without limits—for in what sense is the fraction $\frac{1}{3}$ equal to $\cdot 3333 \ldots\ldots$ or $\sqrt{2}$ equal to another interminable decimal, except as the limits of the two infinite convergent series represented by the decimals? Euclid's definition of equality of ratios

(Book V., Def. V.), is made to include incommensurable ratios by considerations dependent on the method of limits, which also occurs repeatedly in Book XII. In Algebra, as the present writer has endeavoured to shew elsewhere (*Cambridge Mathematical Journal*, Feb., 1852), an exact demonstration of the Binomial Theorem must involve the method of limits. The same remark applies to the operation of equating indeterminate coefficients and the theorem $a^0=1$. Neglect of these considerations involves the writers of some treatises in obscurities, errors, and inconsistencies, which bring to remembrance the supposed common origin of the words "gibberish" and "algebra."*

Throughout the present work, the language of infinites and infinitely small quantities has been carefully avoided, partly because they cannot, except by an inaccuracy of language, be spoken of as really existing magnitudes which may be subjected to analytical operations, partly because the language of the method of limits is equally concise, and is, moreover, exact.

That infinity has a real existence must be admitted; for let us conceive any distance, however great, such that the remotest known star is comparatively near; we cannot say that space terminates at that distance. What is beyond the boundary? A void, perhaps, but still space; so that unless we can conceive the existence of a boundary which includes all space within it, and to which no space is external, we are forced to admit the existence of infinite space. But this admission is altogether different from that which subjects infinity to mathematical operations. How is the infinity thus operated upon to be defined? As a magnitude than which none other is greater? But by hypothesis it is the subject of analytical

* Algebra.—"Some, however, derive it from various other Arabic words, as from Geber, a celebrated philosopher, chemist, and mathematician, to whom they ascribe the invention of this science."—*Hutton's Mathematical Dictionary*. Gibberish.—"It is probably derived from the chemical cant, and originally implied the jargon of Geber and his tribe."—*Johnson's Dictionary*.

operations, and therefore of addition. Add, therefore, some quantity; the result is greater than this infinity, or the definition is contradicted. The truth is, that absolute infinity, such as the infinity of space, cannot be intelligibly conceived on the supposition that anything can be added to it.

Similar considerations apply to infinitely small quantities. There is no difficulty in seeing, that of any kind of magnitude the parts may be diminished infinitely, for, however small a part be taken, it may be divided, and thus smaller parts are taken. If, then, an infinitesimal quantity, the subject of analytical operation, be defined to be a real quantity less than any other, the definition may be readily shewn to be inconsistent with itself.

When, therefore, infinitesimals and infinity are introduced into mathematical operations, they ought to be regarded not as having an absolute existence, but merely as the means of expressing the *limits* to which results approach, as quantities in them are continually increased or diminished.

M. Cournot, in his admirable treatise "*Des Fonctions et du Calcul Infinitesimal*" (Paris, 1841), asserts, indeed, that the infinitesimal method does not merely constitute an ingenious artifice; that it is the expression of the natural mode of generation of physical magnitudes which increase by elements smaller than any finite magnitude. But he does not appear to have anywhere defined what he understands by elements smaller than any finite magnitudes; and without such a definition it is impossible to investigate his proposition accurately. If the words of it be interpreted literally it appears to lead to this dilemma: if the elements be not magnitudes, the addition of them produces no increase—if they be magnitudes, they cannot be less than any finite magnitude; for, being magnitudes, they may be divided into less magnitudes.

With respect to the method of limits, M. Cournot is of opinion that questions must occur in which it is necessary to renounce this method, and to substitute for it in language and in calculations the employment of infinitely small quantities of different orders. He has not, however, specified any instance in which the substitution in question is required.

The following demonstrations do not refer directly or indirectly to different orders of small quantities, nor, indeed, to small quantities at all; for the use of the term "small," in an absolute sense, in mathematics, is objectionable on account of its inexactness. The limit where greatness ceases and smallness begins cannot be distinguished. Hence, though one quantity may be accurately said to be smaller than another, the former cannot with perfect exactness be said to be necessarily and absolutely small with respect to the latter.

The exclusive adherence to the "summatory" definition of the Integral Calculus, has rendered it necessary to present the greater part of the following propositions in a new form, and scarcely anything here given (except the historical notices) is compiled from analogous treatises. The first section contains a popular exposition of the Integral Calculus; and the second a brief account of its history, compiled from one or two cyclopædias and dictionaries. The two following sections are probably in a great measure new, as in them the general principles of integration and the integration of the fundamental functions are derived from the definition above referred to. The three short sections which succeed contain nothing original; but the eighth, on Rational Fractions, is almost entirely newly written. The ordinary demonstration of the possibility of resolving a rational fraction into partial fractions proceeds by the method of equating coefficients, and is defective in this respect—that it neglects to shew, *à priori*, that the assumed coefficients have any real existence, and that the equations determining them do not give impossible or inconsistent results.

To the kindness of Professor De Morgan, of University College, London, the Author is indebted for an exact demonstration of the existence of partial fractions corresponding to rational fractions, with denominators resolvable