

# **ELEMENTS OF QUATERNIONS**

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Elements of Quaternions by A. S. Hardy

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**A. S. HARDY**

**ELEMENTS OF  
QUATERNIONS**



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OF  
QUATERNIONS.

BY

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## PREFACE.

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THE object of the following treatise is to exhibit the elementary principles and notation of the Quaternion Calculus, so as to meet the wants of beginners in the class-room. The *Elements* and *Lectures* of Sir William Rowan Hamilton, while they may be said to contain the suggestion of all that will be done in the way of Quaternion research and application, are not, for this reason, as also on account of their diffuseness of style, suitable for the purposes of elementary instruction. Tait's work on Quaternions is also, in its originality and conciseness, beyond the time and needs of the beginner. In addition to the above, the following works have been consulted:

*Calcolo dei Quaternione.* Bellavitis; Modena, 1858.

*Exposition de la Méthode des Équipollences.* Traduit de l'Italien de Giusto Bellavitis, par C.-A. Laisant; Paris, 1874. (Original memoir in the *Memoirs of the Italian Society.* 1854.)

*Théorie Élémentaire des Quantités Complexes.* J. Houël; Paris, 1874.

*Essai sur une Manière de Représenter les Quantités Imaginaires dans les Construction Géométriques.* Par R. Argand; Paris, 1806. Second edition, with preface

by J. Houël; Paris, 1874. Translated, with notes, from the French, by A. S. Hardy. Van Nostrand's Science Series, No. 52; 1881.

*Kurze Anleitung zum Rechnen mit den (Hamilton'schen) Quaternionen.* J. Odstrčil; Halle, 1879.

*Applications Mécaniques du Calcul des Quaterniones.* Laisant; Paris, 1877.

*Introduction to Quaternions.* Kelland and Tait; London, 1873.

A free use has been made of the examples and exercises of the last work; and, in Article 87, is given, by permission, the substance of a paper from Volume I., page 379, *American Journal of Mathematics*, illustrating admirably the simplicity and brevity of the Quaternion method.

If this presentation of the principles shall afford the undergraduate student a glimpse of this elegant and powerful instrument of analytical research, or lead him to follow their more extended application in the works above cited, the aim of this treatise will have been accomplished.

The author expresses his obligation to Mr. T. W. D. Worthen for valuable assistance in the preparation of this work, and to Mr. J. S. Cushing for whatever of typographical excellence it possesses.

A. S. HARDY.

HANOVER, N.H., June 21, 1881.

# CONTENTS.

## CHAPTER I.

### Addition and Subtraction of Vectors; or, Geometric Addition and Subtraction.

Article.	Page
1. Definition of a <i>vector</i> . Effect of the minus sign before a vector . . . . .	1
2. Equal vectors . . . . .	2
3. Unequal vectors. Vector addition . . . . .	2
4. Vector addition, commutative . . . . .	3
5. Vector addition, associative . . . . .	3
6. Transposition of terms in a vector equation . . . . .	4
7. Definition of a <i>tensor</i> . . . . .	4
8. Definition of a <i>scalar</i> . . . . .	5
9. Distributive law in the multiplication of vector by scalar quantities . . . . .	6
10. If $\Sigma\alpha + \Sigma\beta = 0$ , then $\Sigma\alpha = 0$ and $\Sigma\beta = 0$ . . . . .	7
11. Examples . . . . .	8
12. Coplanar vectors. Condition of coplanarity . . . . .	15
13. Co-initial vectors. Condition of collinearity . . . . .	16
14. Examples . . . . .	17
15. Expression for a medial vector . . . . .	24
16. Expression for an angle-bisector . . . . .	25
17. Examples . . . . .	26
18. Mean point . . . . .	28
19. Examples . . . . .	28
20. Exercises . . . . .	30

## CHAPTER II.

### Multiplication and Division of Vectors; or, Geometric Multiplication and Division.

21. Elements of a <i>quaternion</i> . . . . .	33
22. Equal quaternions . . . . .	34
23. Positive rotation . . . . .	35



Article.	Page.
24. Analytical expression for a quaternion. Product and quotient of rectangular unit-vectors. Tensor and versor of a quaternion . . . . .	36
25. Symbolic notation $q = TqUq$ . . . . .	39
26. Reciprocal of a quaternion . . . . .	39
27. Quadrantal versors, $i, j, k$ . . . . .	40
28. Whole powers of unit vectors. Square of a unit vector is $-1$ , . . . . .	41
29. Associative law in the multiplication of rectangular unit-vectors . . . . .	42
30. Negative sign commutative with $i, j, k$ . . . . .	43
31. Commutative law not true for the products of $i, j, k$ . . . . .	43
32. Reciprocal of a unit vector . . . . .	44
33. A unit vector commutative with its reciprocal. Reciprocal of any vector . . . . .	45
34. Product and quotient of any two rectangular vectors . . . . .	47
35. Square of any vector . . . . .	47
36. Distributive law true of the products of $i, j, k$ . . . . .	48
37. Exercises . . . . .	48
38. Symbolic notation, $q = Sq + Vq$ . . . . .	49
39. De Moivre's theorem . . . . .	50
40. Products of two vectors. Symbolic notation . . . . .	56
41. General principles and formulae . . . . .	58
42. Powers of vectors and quaternions . . . . .	61
43. Relation between the vector and Cartesian determination of a point . . . . .	64
44. Right, coplanar, diplanar and collinear quaternions. Any two quaternions reducible to the forms $\frac{\alpha}{\beta}, \frac{\gamma}{\beta}$ and $\frac{\alpha}{\gamma}, \frac{\gamma}{\delta}$ . . . . .	65
45. Reciprocal of a vector, scalar and quaternion . . . . .	66
46. Conjugate of a vector, scalar and quaternion . . . . .	68
47. Opposite quaternions . . . . .	70
48. $UKq = U \frac{1}{q} = \frac{1}{Uq} = KUq$ . . . . .	71
49. Representation of versors by spherical arcs . . . . .	71
50. Addition and subtraction of quaternions. $K, S$ and $V$ distributive symbols. $K$ commutative with $S$ and $V$ . $T$ and $U$ not distributive symbols . . . . .	72
51. Multiplication of quaternions; not commutative. $UNq = \Pi Uq$ . $Trq = \Pi Tq$ . $Kqr = KrKq$ . Product or quotient of coplanar quaternions . . . . .	74
52. Distributive and associative laws in quaternion and vector multiplication . . . . .	79
53. General formulae . . . . .	82

CONTENTS.

vii

Article.	Page.
54. Applications . . . . .	82
55. Formulae relating to the products of two or more vectors . . . . .	99
56. Exercises . . . . .	106
57. Examples. Applications to Spherical Trigonometry . . . . .	108
58. General Formulae . . . . .	119
59. Applications to Plane Trigonometry . . . . .	125

CHAPTER III.

Applications to Locii.

60. General equations of a line and surface . . . . .	131
61. Use of Cartesian forms in conjunction with quaternion reasoning . . . . .	132
62. Non-commutative law in quaternion differentiation. Differentiation of scalar functions . . . . .	132
63. Quaternion differentiation . . . . .	134
64. Illustration . . . . .	136
65. Distributive principle . . . . .	138
66. Quadrinomial form . . . . .	139
67. Examples . . . . .	140

The Right Line.

68. Right line through the origin . . . . .	145
69. Parallel lines . . . . .	145
70. Right line through two given points . . . . .	146
71. Perpendicular to a given line; its length . . . . .	147
72. Equations of a right line are linear and involve one independent variable scalar . . . . .	150

The Plane.

73. Equation of a plane . . . . .	151
74. Plane making equal angles with three given lines . . . . .	152
75. Plane through three given points . . . . .	153
76. Equations of a plane are linear and involve two independent variable scalars . . . . .	154
77. Exercises and problems on the right line and plane . . . . .	154

The Circle and Sphere.

78. Equations of the circle . . . . .	164
79. Equations of the sphere . . . . .	165

Article.	Page.
80. Tangent line and plane . . . . .	166
81. Chords of contact . . . . .	167
82. Exercises and problems on the circle and the sphere . . . . .	167
88. Exercises in the transformation and interpretation of elementary symbolic forms . . . . .	176

**The Conic Sections. Cartesian Forms.**

84. The parabola . . . . .	178
85. Tangent to the parabola . . . . .	178
86. Examples on the parabola . . . . .	180
87. Relations between three intersecting tangents to the parabola . . . . .	185
88. The ellipse . . . . .	191
89. Examples on the ellipse . . . . .	192
90. The hyperbola . . . . .	195
91. Examples on the hyperbola . . . . .	195

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92. Linear equation in quaternions. Conjugate and self-conjugate functions . . . . .	199
93. General equations of the conic sections . . . . .	201
94. The ellipse . . . . .	204
95. Examples . . . . .	206
96. The parabola . . . . .	214
97. Examples . . . . .	216
98. The cycloid . . . . .	222
99. Elementary applications to mechanics . . . . .	223
100. Miscellaneous Examples . . . . .	231