

**ELEMENTS OF THE  
DIFFERENTIAL CALCULUS:  
WITH EXAMPLES AND  
APPLICATIONS. A TEXT BOOK**

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649571284

Elements of the Differential Calculus: With Examples and Applications. A Text Book by W. E. Byerly

Except for use in any review, the reproduction or utilisation of this work in whole or in part in any form by any electronic, mechanical or other means, now known or hereafter invented, including xerography, photocopying and recording, or in any information storage or retrieval system, is forbidden without the permission of the publisher, Trieste Publishing Pty Ltd, PO Box 1576 Collingwood, Victoria 3066 Australia.

All rights reserved.

Edited by Trieste Publishing Pty Ltd.  
Cover @ 2017

This book is sold subject to the condition that it shall not, by way of trade or otherwise, be lent, re-sold, hired out, or otherwise circulated without the publisher's prior consent in any form or binding or cover other than that in which it is published and without a similar condition including this condition being imposed on the subsequent purchaser.

[www.triestepublishing.com](http://www.triestepublishing.com)

**W. E. BYERLY**

**ELEMENTS OF THE  
DIFFERENTIAL CALCULUS:  
WITH EXAMPLES AND  
APPLICATIONS. A TEXT BOOK**



ELEMENTS  
OF THE  
DIFFERENTIAL CALCULUS,  
WITH  
EXAMPLES AND APPLICATIONS.

A TEXT BOOK

BY

W. E. BYERLY, PH.D.,

ASSISTANT PROFESSOR OF MATHEMATICS IN HARVARD UNIVERSITY.

---

BOSTON, U.S.A.:  
PUBLISHED BY GINN & COMPANY.  
1901.

## PREFACE.

---

THE following book, which embodies the results of my own experience in teaching the Calculus at Cornell and Harvard Universities, is intended for a text-book, and not for an exhaustive treatise.

Its peculiarities are the rigorous use of the Doctrine of Limits as a foundation of the subject, and as preliminary to the adoption of the more direct and practically convenient infinitesimal notation and nomenclature; the early introduction of a few simple formulas and methods for integrating; a rather elaborate treatment of the use of infinitesimals in pure geometry; and the attempt to excite and keep up the interest of the student by bringing in throughout the whole book, and not merely at the end, numerous applications to practical problems in geometry and mechanics.

I am greatly indebted to Prof. J. M. Peirce, from whose lectures I have derived many suggestions, and to the works of Benjamin Peirce, Todhunter, Duhamel, and Bertrand, upon which I have drawn freely.

W. E. BYERLY.

CAMBRIDGE, October, 1879.



## TABLE OF CONTENTS.

### CHAPTER I.

#### INTRODUCTION.

Article.	Page
1. Definition of <i>variable</i> and <i>constant</i> . . . . .	1
2. Definition of <i>function</i> and <i>independent variable</i> . . . . .	1
3. Symbols by which functional dependence is expressed . . . . .	2
4. Definition of <i>increment</i> . Notation for an increment. An increment may be positive or negative . . . . .	2
5. Definition of the <i>limit</i> of a variable . . . . .	3
6. Examples of <i>limits</i> in Algebra . . . . .	3
7. Examples of <i>limits</i> in Geometry . . . . .	4
8. The fundamental proposition in the <i>Theory of Limits</i> . . . . .	5
9. Application to the proof of the theorem that the area of a circle is one-half the product of the circumference by the radius. . . . .	5
10. Importance of the clear conception of a <i>limit</i> . . . . .	6
11. The velocity of a moving body. <i>Mean velocity</i> ; <i>actual velocity</i> at any instant; <i>uniform velocity</i> ; <i>variable velocity</i> . . . . .	6
12. Actual velocity easily indicated by aid of the <i>increment</i> notation . . . . .	7
13. Velocity of a falling body . . . . .	7
14. The direction of the tangent at any point of a given curve. Definition of <i>tangent</i> as limiting case of <i>secant</i> . . . . .	8
15. The <i>inclination of a curve to the axis of X</i> easily indicated by the aid of the <i>increment</i> notation . . . . .	8
16. The inclination of a parabola to the axis of <i>X</i> . . . . .	9
17. Fundamental object of the Differential Calculus . . . . .	10

### CHAPTER II.

#### DIFFERENTIATION OF ALGEBRAIC FUNCTIONS.

18. Definition of <i>derivative</i> . Derivative of a <i>constant</i> . . . . .	11
19. General method of finding the derivative of any given function. General formula for a derivative. Examples . . . . .	11



Article.	Page.
20. Classification of functions . . . . .	12
21. Differentiation of the product of a constant and the variable; of a power of the variable, where the exponent is a positive integer . . . . .	13
22. Derivative of a sum of functions . . . . .	14
23. Derivative of a product of functions . . . . .	15
24. Derivative of a quotient of functions. Examples . . . . .	17
25. Derivative of a function of a function of the variable . . . . .	18
26. Derivative of a power of the variable where the exponent is negative or fractional. Complete set of formulas for the differentiation of Algebraic functions. Examples . . . . .	19

## CHAPTER III.

## APPLICATIONS.

*Tangents and Normals.*

27. Direction of tangent and normal to a plane curve . . . . .	22
28. Equations of tangent and normal. Subtangent. Subnormal. Length of tangent. Length of normal. Examples . . . . .	23
29. Derivative may sometimes be found by solving an equation. Examples . . . . .	25

*Indeterminate Forms.*

30. Definition of infinite and infinitely great . . . . .	26
31. Value of a function corresponding to an infinite value of the variable . . . . .	26
32. Infinite value of a function corresponding to a particular value of the variable . . . . .	27
33. The expressions $\frac{0}{0}$ , $\frac{\infty}{\infty}$ , and $0 \times \infty$ , called indeterminate forms. When definite values can be attached to them . . . . .	28
34. Treatment of the form $\frac{0}{0}$ . Examples . . . . .	28
35. Reduction of the forms $\frac{\infty}{\infty}$ and $0 \times \infty$ to the form $\frac{0}{0}$ . . . . .	30

*Maxima and Minima of a Continuous Function.*

36. Continuous change. Continuous function . . . . .	31
37. If a function increases with the increase of the variable, its derivative is positive; if it decreases, negative . . . . .	31
38. Value of derivative shows rate of increase of function . . . . .	32
39. Definition of maximum and minimum values of a function . . . . .	32

TABLE OF CONTENTS.

vii

Articles.	Page.
40. <i>Derivative zero at a maximum or a minimum</i> . . . . .	33
41. Geometrical illustration . . . . .	33
42. Sign of derivative near a zero value shown by the value of its own derivative . . . . .	34
43. Derivatives of different orders . . . . .	34
44. Numerical example . . . . .	34
45. Investigation of a minimum . . . . .	35
46. Case where the third derivative must be used. Examples . . . . .	35
47. General rule for discovering maxima and minima. Examples . . . . .	36
48. Use of auxiliary variables. Examples . . . . .	38
49. Examples . . . . .	39

*Integration.*

50. Statement of the problem of finding the <i>distance</i> traversed by a falling body, given the <i>velocity</i> . . . . .	41
51. Statement of the problem of finding the <i>area</i> bounded by a given curve . . . . .	41
52. Statement of the problem of finding the length of an arc of a given curve . . . . .	42
53. <i>Integration. Integral</i> . . . . .	44
54. <i>Arbitrary constant</i> in integration . . . . .	44
55. Some formulas for direct integration . . . . .	44
56. Solution of problem stated in Article 50 . . . . .	45
57. Example under problem stated in Article 51. Examples . . . . .	46
58. Examples under problem stated in Article 52 . . . . .	48

CHAPTER IV.

TRANSCENDENTAL FUNCTIONS.

59. Differentiation of $\log x$ requires the investigation of the limit of $\left(1 + \frac{1}{m}\right)^m$ . . . . .	49
60. Expansion of $\left(1 + \frac{1}{m}\right)^m$ by the Binomial Theorem . . . . .	50
61. Proof that the limit in question is the sum of a well-known series . . . . .	50
62. This series is taken as the base of the natural system of logarithms. Computation of its numerical value . . . . .	52
63. Extension of the proof given above to the cases where $m$ is not a positive integer . . . . .	53
64. Differentiation of $\log x$ completed . . . . .	54
65. Differentiation of $a^x$ . Examples . . . . .	55

Article.	Trigonometric Functions.	Page
66.	Circular measure of an angle. Reduction from <i>degree</i> to circular measure. Value of the <i>unit</i> in circular measure . . .	57
67.	Differentiation of $\sin z$ requires the investigation of the limit $\frac{\sin \Delta x}{\Delta x}$ and $\frac{1 - \cos \Delta x}{\Delta x}$ . . . . .	57
68.	Investigation of these limits . . . . .	58
69.	Differentiation of the Trigonometric Functions. Examples . . .	59
70.	<i>Anti-</i> or <i>inverse</i> Trigonometric Functions . . . . .	60
71.	Differentiation of the <i>Anti-Trigonometric</i> Functions. Examples . . .	60
72.	<i>Anti-</i> or <i>inverse</i> notation. Differentiation of <i>anti-</i> functions in general . . . . .	61
73.	The derivative of $y$ with respect to $x$ , and the derivative of $x$ with respect to $y$ , are reciprocals. Examples . . . . .	62

## CHAPTER V.

## INTEGRATION.

74.	Formulas for <i>direct</i> integration . . . . .	65
75.	Integration by <i>substitution</i> . Examples . . . . .	66
76.	If $fx$ can be integrated, $f(a + bx)$ can always be integrated. Examples . . . . .	67
77.	$\int_a \frac{1}{\sqrt{(a^2 - x^2)}} dx$ . Examples . . . . .	67
78.	$\int_a \frac{1}{\sqrt{(a^2 + x^2)}} dx$ . Example . . . . .	68
79.	<i>Integration by parts</i> . Examples . . . . .	69
80.	$\int_a \sin^2 x$ . Examples . . . . .	69
81.	Use of <i>integration by substitution</i> and <i>integration by parts</i> in combination. Examples . . . . .	70
82.	Simplification by an <i>algebraic transformation</i> . Examples . . . . .	71

*Applications.*

83.	Area of a segment of a circle; of an ellipse; of an hyperbola . . .	72
84.	Length of an arc of a circle . . . . .	74
85.	Length of an arc of a parabola. Example . . . . .	75

## CHAPTER VI.

## CURVATURE.

86.	Total curvature; mean curvature; actual curvature. Formula for actual curvature . . . . .	77
-----	---	----