SECULAR PERTURBATIONS ARISING FROM THE ACTION OF JUPITER ON MARS

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ARTHUR BERTRAM TURNER

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SECULAR PERTURBATIONS

ARISING FROM THE

ACTION OF JUPITER ON MARS

A THESIS

PRESENTED TO THE FACULTY OF PHILOSOPHY OF THE UNIVERSITY
OF PENNSYLVANIA

BY

ARTHUR BERTRAM TURNER

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CONTENTS.

I.

Lagrange's Generalized Equations of Motion. Lagrange's Canonical Equations.

Let $F_{11}, F_{12}, F_{13}, \cdots, F_{1n}$ be the forces acting on a unit of mass m_1,

$F_{21}, F_2F_2, F_2F_3, \cdots, F_{2n}$ be the forces acting on a unit of mass m_2,

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

etc. $\qquad\qquad\qquad\qquad$ etc.

Let $\delta p_{11}, \delta p_{12}, \delta p_{13}, \cdots, \delta p_{1n}$ be the virtual velocities of m_1,

$\delta p_{21}, \delta p_{22}, \delta p_{23}, \cdots, \delta p_{2n}$ be the virtual velocities of m_2,

$\vdots$ $\qquad\qquad\qquad\qquad$ $\vdots$

etc. $\qquad\qquad\qquad\qquad$ etc.

Now assume that each mass m_i be displaced an infinitesimal distance $l = ds_i$ in the direction in which the mass m_i would have moved during the next instant had it not been subjected to this arbitrary displacement, and let the distance in each case be precisely equal to the distance which the body would have moved during the next instant had it not been subjected to displacement. Then by the theorem in virtual velocities that $\sum F\delta p = \delta t =$ change in the living force, we shall have for the masses $m_1 \cdots m_k$,

$$k\sum_1^n m_1 F_{1k}\delta p_{1k} = \delta T \quad \text{for} \quad m_1,$$
$$\vdots \qquad\qquad \vdots$$
$$k\sum_1^n m_k F_{ik}\delta p_{ik} = \delta T \quad \text{for} \quad m_k,$$

adding we get

(a) $$\delta T = i\sum_1^k k\sum_1^n m_i F_{ik}\delta p_{ik}.$$

These equations involve the masses because F_{ik} are forces on unit mass.

Now it is known that the change in the living force of a system is equal to the work done on the system and since work equals force × distance, we shall get for the change in the living force

$$\text{(b)} \qquad \delta T = i\sum_{1}^{k} m_i \frac{d^2 s_i}{dt^2}\delta s_i$$

Equating these two values of δT, we get,

$$\text{(1)} \qquad i\sum_{1}^{k}\left\{ k\sum_{1}^{n} m_i F_{ik}\delta p_{ik} - m_i \frac{d^2 s_i}{dt^2}\delta s_i \right\} = 0$$

which is Lagrange's Generalized Equation.

If now we suppose the forces to be resolved along the three coördinate axes the above equation can be easily made to assume the form,

$$\text{(2)} \qquad \sum\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \sum\left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \sum\left(Z - m\frac{d^2z}{dt^2}\right)\delta z = 0$$

where X, Y, Z are the total components of the forces along the coördinate axes.

Let us assume a certain function U (Potential Function) which is independent of the time t, such that

$$\frac{\partial U}{\partial x} = X, \qquad \frac{\partial U}{\partial y} = Y, \qquad \frac{\partial U}{\partial z} = Z;$$

then by substitution equation (2) becomes

$$\sum\left(\frac{\partial U}{\partial x}\delta x + \cdots \text{etc.}\right) = \sum\left(m\frac{d^2x}{dt^2}\delta x + \cdots \text{etc.}\right).$$

Now the left hand member of this equation is the total variation of U, or δU.

Since T (Living Force) $= \frac{1}{2}mv^2$, $\delta T = mv\delta v$, but

$$m \frac{d^2x}{dt^2}\delta x = m\frac{dv}{dt}\delta x,$$

and adding

$$m \frac{d^2x}{dt^2}\delta x = m\frac{dv}{dt}\delta x + mv\delta v - \delta T$$

now

$$\frac{d}{dt}(mv\delta x) = m\frac{dv}{dt}\delta x + mv\frac{d}{dt}(\delta x) = m\frac{dv}{dt}\delta x + mvdv,$$

for

$$mv\frac{d}{dt}(\delta x) = mv\delta\left(\frac{dx}{dt}\right) = mv\delta v.$$

Hence

$$m\frac{d^2x}{dt^2}\delta x = \frac{d}{dt}(mv\delta x) - \delta T$$

$$\therefore \sum\left(m\frac{d^2x}{dt}\delta x + \cdots \text{etc.}\right) = \frac{d}{dt}(mv\delta s) - \delta T,$$

or

$$(8) \qquad \delta U = \frac{d}{dt}(mv\delta s) - \delta T.$$

Let us suppose T to be a function of the independent variables $q_1, q_2, \cdots$, etc., then the variation of T is

$$\delta T = \frac{\partial T}{\partial q_1}\delta q_1 + \cdots \text{etc.},$$

$$\delta U = \frac{\partial U}{\partial q_1}\delta q_1 + \cdots \text{etc.},$$

$$\delta s = \frac{\partial s}{\partial q_1}\delta q_1 + \cdots \text{etc.}$$

These values substituted in (8) give the equation

$$\left(\frac{\partial U}{\partial q_1}\delta q_1 + \cdots \text{etc.}\right) = \frac{d}{dt}\left(mv\left[\frac{\partial s}{\partial q_1}\delta q_1 + \cdots \text{etc.}\right]\right) - \left(\frac{\partial T}{\partial q_1}\delta q_1 + \cdots \text{etc.}\right)$$

and since the q's are independent we can equate the like variations and obtain the following partial differential equations:—

$$\frac{\partial U}{\partial q_1} = \frac{d}{dt}\left(m\dot{v}\frac{\partial s}{\partial q_1}\right) - \frac{\partial T}{\partial q_1}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

which become

$$\frac{\partial U}{\partial q_1} = \frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} \tag{4}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\text{etc.} \qquad \text{etc.} \qquad \text{etc.}$$

Since

$$v = \frac{ds}{dt} = \sum \frac{\partial s}{\partial q}\cdot\frac{dq}{dt} + \frac{\partial s}{\partial t} \quad \text{and} \quad \frac{\partial v}{\partial q_1'} = \frac{\partial s}{\partial q_1}.$$

But $\frac{1}{2}mv^2 = T$, therefore

$$\frac{\partial T}{\partial q_1'} = mv\frac{\partial v}{\partial q_1'} = mv\frac{\partial s}{\partial q_1}.$$

These equations are known as Lagrange's Canonical Forms, and in deriving them we have assumed that all points of the system have been expressed in terms of t, and k independent variables $q_1 \cdots q_k$. Since there are $3n$ coördinates altogether in the system, $(x_1\, y_1\, z_1, \cdots, x_n\, y_n\, z_n)$ this assumes that there are $(3n - k)$ equations of condition.

II.

Canonical Forms of Hamilton.

Let us still regard T as expressed in terms of $q, \cdots, q_k$, $q_1', \cdots, q_k'$, and write

$$p_1 = \frac{\partial T}{\partial q_1'}, \quad p_2 = \frac{\partial T}{\partial q_2'}, \quad \cdots, \quad \text{etc}.$$

T was originally a homogeneous function in regard to

$$\frac{dx_1}{dt}, \quad \frac{dx_2}{dt}, \quad \cdots$$