STUDIES ON DIVERGENT SERIES AND SUMMABILITY; MICHIGAN SCIENCE SERIES - VOL. II

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WALTER BURTON FORD

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BY

WALTER BURTON FORD, Ph.D.

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PREFACE

During the academic year 1908–9 the author was privileged to give as a part of his work at the University of Michigan a course of lectures on infinite series, with especial reference in the second semester to *divergent* series — a subject which, despite the uncertain value so long attached to it, seemed clearly to be coming into increasing prominence and importance in mathematical analysis. Little was accomplished, however, as regards divergent series beyond the merest beginning; yet this was sufficient to awaken a desire to continue farther and this in turn resulted in a course being given throughout the whole of the following year devoted entirely to divergent series and the related topic of summability. But this year also closed with much less ground satisfactorily covered than had been expected, unforeseen difficulties having arisen from time to time, some due to the inherent complexities of the subject in hand and others to the somewhat hastily conceived and hence unsatisfactory state in which much of the related literature was found to be. Thus the course still seemed altogether incomplete. It was therefore decided to continue it once more throughout the following year, 1910–11, and indeed for a like reason it was finally continued throughout 1911–12. As the lectures and class-room discussions progressed, permanent notes were kept in the hope that the whole might possibly pass through the press at some future time and appear in book form—a hope which, after various delays during which the original notes have been considerably supplemented, now reaches its realization in the appearance of the present volume. In its final form it certainly presents a large mass of detail and is doubtless open to criticism in many respects, but it does not seem advisable to attempt any further defence for it than is contained in the remaining sections of this preface wherein, after certain generalities, the content and motive of the various chapters are discussed in some detail.

Speaking roughly, the study of divergent series, at least as the author has come to conceive of it, may be divided into two parts, the one concerning the so-called *asymptotic series* and the other the *theory of summability*. Of these the first, representing the older aspect, originated in an isolated note by Cauchy in 1843[1] relating to the well-known series of Stirling for $\log \Gamma(x)$, viz.:

$$(1)\quad \log \Gamma(x) = \tfrac{1}{2}\log 2\pi + (x - \tfrac{1}{2})\log x - x + \frac{B_1}{1\cdot 2}\frac{1}{x} - \frac{B_2}{3\cdot 4}\frac{1}{x^3} + \frac{B_3}{5\cdot 6}\frac{1}{x^5} - \cdots.$$

(B_m = mth Bernoulli number.)

Cauchy pointed out that this series, though divergent for all values of x, may be

[1] "Sur l'emploi légitime des séries divergentes," *Compt. Rend. de l'Acad. des Sciences*, Vol. 17, pp. 370–376.

used in computing $\log \Gamma(x)$ when x is large (and positive)—in fact, it was shown that, having fixed the number n of terms taken, the absolute error committed by stopping the summation at the nth term is less than the absolute value of the next succeeding term, and hence becomes arbitrarily small ($n > 3$) with increasing x. CAUCHY's work on divergent series was confined, however, to the single series (1) and, owing to the emphasis placed upon convergent processes exclusively by the successors of CAUCHY and ABEL, no further progress was made in this interesting field until the subject at last reappeared after more than forty years in connection with the researches of POINCARÉ upon the irregular solutions of linear differential equations.[2] POINCARÉ considered those divergent series (normal series) of the form

$$(2)\qquad e^{f(x)}x^{\rho}(a_0 + a_1/x + a_2/x^2 + \cdots); \qquad \begin{array}{l} f(x) = \textit{polynomial in } x, \\ \rho = \textit{constant}. \end{array}$$

which for some time had been known to satisfy *formally* linear differential equations of certain types having the point $x = \infty$ as an "irregular" point, and he showed essentially that in general to every such formal solution there corresponds an *actual* solution which can be represented by (2) in much the same sense as (1) was described above as representing $\log \Gamma(x)$.[3] In view of the important significance of such results both from the standpoint of the possible use of divergent series as well as from that of the theory of differential equations, POINCARÉ set apart and discussed in some detail a broad class of divergent series of the special form (2), applying to them the name of "asymptotic series." POINCARÉ's results, however, in so far as they concerned differential equations, were noticeably incomplete, being limited by certain unfortunate restrictions, and thus his original studies have given rise in later years to numerous researches, notably by HORN, in which noteworthy advances have been made, though open questions in this connection still remain. Corresponding investigations (likewise begun by POINCARÉ) pertaining to linear difference equations have been undertaken in recent years and carried to an advanced stage by HORN, NÖRLUND, and others. Meanwhile an important aspect of the theory of asymptotic series has come into view, especially in England under the leadership of BARNES and HARDY; namely, that of actually determining the asymptotic developments of a given function—a problem of decided interest for the study and classification of functions in general. This latter aspect of the subject presents a high degree of complexity and doubtless has made hardly more than a beginning at the present time. In fact, it has thus far been approached only by confining the attention to a very limited number of special functional types.[4]

[2] "Sur les intégrales irrégulières des équations linéaires," *Acta Math.*, Vol. 8 (1886), pp. 259–344. Mention should be made also of STIELTJES who simultaneously with POINCARÉ resumed the study of divergent series, confining his attention, however, to the computational aspects of certain special series. (Thesis, *Ann. de l'Éc. Nor.* (3), Vol. 3 (1886), p. 201.)

[3] For the more accurate statements, see Chap. III.

[4] For details, see Chap. II.

The theory of summability, or second general aspect of divergent series mentioned above, is essentially concerned with the question as to whether in any proper sense a " sum " may be assigned to the series, assumed divergent,

$$\sum_{n=0}^{\infty} a_n. \tag{3}$$

This question has been scientifically attacked only within comparatively recent years, the most common avenue of approach being through the so-called boundary-value (Grenzwert) problem in the theory of analytic functions.[5] Thus FROBENIUS, without having in view the study of divergent series, showed in the first place that if one has a power series whose radius of convergence is equal to 1:

$$\sum_{n=0}^{\infty} a_n x^n; \quad r = \textit{radius of convergence} = 1 \tag{4}$$

and writes $s_n = a_0 + a_1 + \cdots + a_n$, then

$$\lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n = \lim_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1} \tag{5}$$

whenever the indicated limit on the right exists.[6] Now, the first member of (5) is naturally associated with the corresponding series (3) (in general divergent) obtained by placing $x = 1$ in (4). Thus, at least if one confines the attention to divergent series (3) of the particular type just mentioned, it becomes natural to assign sums in accordance with the formula

$$s = \lim_{n=\infty} \frac{s_0 + s_1 + \cdots + s_n}{n+1}, \tag{6}$$

whenever the indicated limit exists. Moreover, this formula finds additional justification in the demonstrable fact that for any *convergent* series (3) the sum, regarded in the ordinary sense, viz., $s = \lim_{n=\infty} s_n$, agrees with that given by (6)—*i. e.*, formula (6) is *consistent*. Aside from this one formula (6) many others are now known which serve with more or less appropriateness to define the sum of a divergent series, both when the series is of the special type above mentioned and when otherwise. To what ultimate extent these formulas *are* appropriate, how far the theories of summability erected upon them serve any justifiable purpose in analysis, whether the different sums thus assigned involve mutual inconsistencies—these and other questions may well be asked and more will be said on this point presently.[7] Suffice it to say here that formula (6) has been found in

[5] For an elementary description of the problem, see JAHRAUS, "Das Verhalten der Potenzreihen auf dem Konvergenzkreise historisch-kritisch dargestellt." Program des Gymnasiums Ludwigshafen (1901), pp. 1–56. See also KNOPP, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze." Dissertation, Berlin, 1907.

[6] "Ueber die Leibnitzsche Reihe," *Jour. für Math.*, Vol. 89 (1880), pp. 262–264.

[7] Interesting comments by PRINGSHEIM relative to such questions are to be found in Vol. I of the "Encyklopädie der math. Wissenschaften," §§ 39–40.

particular to yield interesting and valuable results when applied to Fourier series and the other important allied developments in mathematical physics — developments in terms of Bessel functions, Legendre functions, etc. Such applications alone go far toward assuring a permanent place in analysis to the theory of summability as now commonly understood.

Turning now more specifically to the contents of the present volume, Chapter I considers certain aspects of the so-called Maclaurin Sum-Formula, the especial aim being to develop and summarize into actual theorems those results which are of importance in this connection to the study of divergent series. These when once obtained are of particular service in the problem of determining the asymptotic developments of a given function, and it is to this that Chapter II is then devoted. Beginning with very easy illustrative studies, the Chapter proceeds to problems of greater and greater difficulty and eventually treats the general problem already considered by various investigators of determining the asymptotic developments of the general integral (entire) function of rank p (order > 0), following which, at the close of the chapter, the problem of determining the asymptotic developments of functions defined by power series is briefly considered. Chapter III concerns the asymptotic solutions of linear differential equations and is an attempt to summarize briefly and without proof what are deemed to be the most essential results thus far known in this field, with mention also of the corresponding results obtainable in the study of linear difference equations, and with indications as to certain open questions still remaining in both connections. Chapter IV considers the theory of summability with the especial attempt, as in previous chapters, to single out what seems most essential. More specifically, it makes an examination of a few of the standard definitions of " sum " with the idea of subjecting each to a number of tests which, as the author has come to view the subject, every such definition should satisfy. For example, it is well known that if a really logical general theory of summability is ever to be constructed it cannot include all definitions of sum that satisfy merely the condition of *consistency* (§ 37) since this alone does not insure uniqueness of sum. Therefore, observing the genesis of the whole subject from the boundary value problem as described above, it is proposed to arbitrarily limit the general theory to those series (3) for which the corresponding power series (4) has a radius of convergence equal to 1 and then retain only such definitions of sum as give the *unique* value

$$s = \lim_{x=1-0} \sum_{n=0}^{\infty} a_n x^n.$$

Definitions which do this are said to satisfy the *boundary value condition* (§ 39). Such definitions not only all give the same sum to a given series (convergent or divergent) (3), but they at once serve a useful purpose in analysis from the fact that they frequently come to furnish the analytic continuation of the series (4) over some portion of its circle of convergence, or indeed in some cases, as in the definitions of BOREL, throughout regions lying entirely outside that circle. However limited the scope of a general theory of summability as thus conceived, it at least has perfect definiteness and logical coherence and finds immediate usefulness in the theory of functions of a complex variable, and we venture the opinion that some such characteristics as these must be preserved in any general theory of summability that is to retain a permanent place in analysis.[8] No attempt will be made here to describe the other tests which Chapter IV sets up, but it should be remarked that only a few of the standard definitions of sum are tested out since they suffice to illustrate the spirit of the undertaking. The chapter closes with a brief account of absolutely summable series and a statement of certain supplementary theorems and corollaries upon summability in general.

A most important aspect of the theory of summability, as the author regards it, lies in its applications mentioned above to Fourier series and other allied developments in mathematical physics, and this forms the subject of Chapter V. For the sake of completeness the treatment is made to include both convergence and summability. It is based upon a general method for the study of all such developments due to DINI and appearing, though in somewhat diffuse and inaccessible form, in his great work entitled "Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale" (Pisa, 1880). DINI naturally considered at the time of his investigations only the question of convergence (not including uniform convergence), but his methods are here shown to be readily extended so as to be applicable to studies in summability. Especial effort has been made here as in the other chapters to summarize all essential conclusions from time to time into actual theorems.

To Professor Alexander Ziwet the author would here express his deep gratitude. Not only has the book enjoyed the benefits of his critical judgment in many ways, but his sympathy and kindly interest have served as a constant encouragement, and indeed they are responsible in no small measure for the appearance of the whole in its present form. The author is much indebted also to his colleagues Professors C. E. Love and Tomlinson Fort, the former for various suggestions and criticisms, and the latter for the valuable aid he has rendered in reading the proofs.

ANN ARBOR,
April, 1915

[8] The adoption of any one definition for "summable series" evidently involves the excluding of many series previously classed as summable; yet we believe the time has arrived when a single universal definition should if possible be agreed upon, however disastrous its immediate effects upon one or more of the special forms of definition now current. The present situation in this matter is strikingly analogous to the state of confusion which led CAUCHY and ABEL to the formulation of their universal definition for "convergent series," notwithstanding the exclusions brought about and the consequent objections urged by contemporary mathematicians.