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THE
MESSENGER OF MATHEMATICS.

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Erratum—p. 46, last line, omit "= 1."

MESSENGER OF MATHEMATICS.

ON CERTAIN SOLUTIONS OF MAXWELL'S EQUATIONS.

By *H. Bateman*.

Vector-fields with moving singular curves.

§ 1. I HAVE shown elsewhere* that it is possible to obtain a solution of Maxwell's equations which represents a vector field in which the electric and magnetic intensities are infinite at a moving point Q, whose coordinates at time α are ξ, η, ζ, and also along a moving curve attached to this point; the curve being the locus of a series of points projected from the different positions of Q, and travelling along straight lines with the velocity of light. The direction of projection for any position of Q was chosen so that it made an angle θ with the tangent to the path of Q such that $c \cos\theta = v$, where v is the velocity of Q, and c the velocity of light. This condition is, however, not invariant under the transformations of the theory of relativity, and I now find that it is not necessary to restrict the direction of projection in the way described; the introduction of the restriction was due to the mistaken idea that the second of equations (291)† is a consequence of the first.

Let α and β be defined as before by the equations

$$(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2=c^2(t-\tau)^2 \quad \ldots\ldots(1),$$

$$l(x-\xi)+m(y-\eta)+n(z-\zeta)=c^2p(t-\tau) \ldots\ldots(2),$$

* *The Mathematical Analysis of Electrical and Optical Wave Motion on the basis of Maxwell's Equations*, p. 128. This will be cited later as E.

† E, p. 129. The error in the proof occurs when the axis of y is chosen so that $n_1=0$; this introduces a restriction, since l_1, m_1, n_1 are generally complex. The same error occurs in one of my previous papers. *Annals of Mathematics* (1914).

where ξ, η, ζ, τ are functions of α only, and l, m, n, p are functions of α and β which depend linearly on β, so that

$$l=\beta l_1-l_0,\quad m=\beta m_1-m_0,\quad n=\beta n_1-n_0,\quad p=\beta p_1-p_0 \ldots (3).$$

To make the values of α and β unique, we write $\tau=\alpha$ and introduce the inequality $\tau \leq t$. The quantities l, m, n, p must, moreover, be chosen so that $l^2+m^2+n^2=c^2p^2$, and so we have the relations

$$l_1^2+m_1^2+n_1^2=c^2p_1^2,\quad l_0l_1+m_0m_1+n_0n_1=c^2p_0p_1,\quad l_0^2+m_0^2+n_0^2=c^2p_0^2 \ldots (4).$$

We now use the symbol f to denote an arbitrary function of α and β, and write

$$P=\xi'(x-\xi)+\eta'(y-\eta)+\zeta'(z-\zeta)-c^2(t-\alpha) \ldots (5).$$

The vector field which will be the subject of discussion is that defined by the electro-magnetic potentials

$$A_x=\frac{lf}{P},\quad A_y=\frac{mf}{P},\quad A_z=\frac{nf}{P},\quad \Phi=\frac{cpf}{P} \ldots\ldots\ldots (6).$$

These have been shown to be wave-functions which satisfy the relation

$$\operatorname{div} A+\frac{1}{c}\frac{\partial\Phi}{\partial t}=0.$$

On calculating the components of the electric and magnetic intensities with the aid of the relations

$$H=\operatorname{rot} A,\quad E=-\frac{1}{c}\frac{\partial A}{\partial t}-\frac{\partial\Phi}{\partial x},$$

we find, as before, that the component of the electric intensity along the radius from ξ, η, ζ, α to x, y, z, t is

$$-\frac{f}{P^2}\left[c^2p-l\xi'-m\eta'-n\zeta'\right].$$

To make the electric charge associated with the singularity ξ, η, ζ, α a constant quantity 4π, we choose f so that

$$f(c^2p-l\xi'-m\eta'-n\zeta')=\xi'^2+\eta'^2+\zeta'^2-c^2.$$

It will be convenient, however, to have a value of f independent of β, and so we shall introduce the condition

$$c^2p_1-l_1\xi'-m_1\eta'-n_1\zeta'=0\,;$$

as before, the value of f is then given by the equation*

$$f(c^2 p_0 - l_0\xi' - m_0\eta' - n_0\zeta') = c^2 - \xi'^2 - \eta'^2 - \zeta'^2 \quad \ldots.(7).$$

We shall suppose that l_0, m_0, n_0, p_0 are real, then f is real and l_1, m_1, n_1, p_1 are generally complex quantities. Let $\bar{l}_1$, $\bar{m}_1$, $\bar{n}_1$, $\bar{p}_1$ be the conjugate complex quantities, then we have the relations

$$\left.\begin{array}{c} \bar{l}_1^2 + \bar{m}_1^2 + \bar{n}_1^2 = c^2\bar{p}_1^2, \quad l_0\bar{l}_1 + m_0\bar{m}_1 + n_0\bar{n}_1 = c^2 p_0\bar{p}_1 \\ c^2\bar{p}_1 - \bar{l}_1\xi' - \bar{m}_1\eta' - \bar{n}_1\zeta' = 0 \end{array}\right\} \ldots(8).$$

Now let a set of real quantities $\bar{l}_0$, $\bar{m}_0$, $\bar{n}_0$, $\bar{p}_0$ be chosen so that

$$\left.\begin{array}{l} \bar{l}_0 l_1 + \bar{m}_0 m_1 + \bar{n}_0 n_1 = c^2\bar{p}_0 p_1, \quad \bar{l}_0\bar{l}_1 + \bar{m}_0\bar{m}_1 + \bar{n}_0\bar{n}_1 = c^2\bar{p}_0\bar{p}_1 \\ \bar{l}_0^2 + \bar{m}_0^2 + \bar{n}_0^2 = c^2\bar{p}_0^2, \quad l_0\bar{l}_0 + m_0\bar{m}_0 + n_0\bar{n}_0 - c^2 p_0\bar{p}_0 = h \neq 0 \end{array}\right\} \ldots(9).$$

Then if we write $x - \xi = X$, $y - \eta = Y$, $z - \zeta = Z$, $t - \alpha = T$,

$$\left.\begin{array}{ll} c^2 p_1 T - l_1 X - m_1 Y - n_1 Z = S, & c^2\bar{p}_1 T - \bar{l}_1 X - \bar{m}_1 Y - \bar{n}_1 Z = \bar{S} \\ c^2 p_0 T - l_0 X - m_0 Y - n_0 Z = U, & c^2\bar{p}_0 T - \bar{l}_0 X - \bar{m}_0 Y - \bar{n}_0 Z = \bar{U} \end{array}\right\} \ldots\ldots(10),$$

we find that, if h is suitably chosen, there is an identity of the type

$$S\bar{S} - U\bar{U} \equiv k(c^2T^2 - X^2 - Y^2 - Z^2) = 0 \ldots\ldots.(11),$$

where k is a function of α whose value may be determined by replacing T by 1, X by ξ', Y by η', and Z by ζ' in the identity. We thus find that

$$fk = \bar{l}_0\xi' + \bar{m}_0\eta' + \bar{n}_0\zeta' - c^2\bar{p}_0 \ldots\ldots\ldots\ldots(12).$$

By considering the relations satisfied by l_1, m_1, n_1, p_1, we see that

$$\xi' = \lambda l_0 + \mu\bar{l}_0,\ \eta' = \lambda m_0 + \mu\bar{m}_0,\ \zeta' = \lambda n_0 + \mu\bar{n}_0,\ 1 = \lambda p_0 + \mu\bar{p}_0 \ldots(13),$$

where λ and μ are quantities to be determined. We deduce at once from these equations that

$$l_0\xi' + m_0\eta' + n_0\zeta' - c^2p_0 = \mu h, \quad \bar{l}_0\xi' + \bar{m}_0\eta' + \bar{n}_0\zeta' - c^2\bar{p}_0 = \lambda h,$$

$$\xi'^2 + \eta'^2 + \zeta'^2 - c^2 = 2\lambda\mu h.$$

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* It should be noticed that when this condition is satisfied the field specified by potentials of type $A_x^0 = \frac{\eta'}{\rho} + \frac{\xi'}{\rho}$ is conjugate to the field specified by Liénard's potentials of type $A_x' = \frac{\xi'}{\rho}$ and the relation $A_x^0 A_x' + A_y^0 A_y' + A_z^0 A_z' - \Phi^0\Phi' = 0$ is satisfied. Compare this with the remark E, p. 135.

Hence
$$2\lambda = \frac{\xi'^2 + \eta'^2 + \zeta'^2 - c^2}{l_0\xi' + m_0\eta' + n_0\zeta' - c^2 p_0} = f \quad \ldots\ldots\ldots\ldots (14).$$

The expression for A_x can now be thrown into a more convenient form. Differentiating equation (2) we obtain

$$l = \left(\beta \frac{\partial S}{\partial \alpha} - \frac{\partial U}{\partial \alpha}\right)\frac{\partial \alpha}{\partial x} + S\frac{\partial \beta}{\partial x};$$

also $\beta S = U$, hence we may write

$$A_x = \frac{f}{P}\frac{\partial \alpha}{\partial x}\left[\frac{U}{S}\frac{\partial S}{\partial \alpha} - \frac{\partial U}{\partial \alpha}\right] + \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{U}{S}\right) \quad \ldots (15).$$

Now let $\bar{A}_x$ be the complex quantity conjugate to A_x, then

$$\bar{A}_x = \frac{f}{P}\frac{\partial \alpha}{\partial x}\left[\frac{\bar{U}}{\bar{S}}\frac{\partial \bar{S}}{\partial \alpha} - \frac{\partial \bar{U}}{\partial \alpha}\right] + \frac{\bar{U}f}{P}\frac{\partial}{\partial x}\log\left(\frac{\bar{U}}{\bar{S}}\right).$$

and so if $2a_x = A_x + \bar{A}_x$, we have

$$2a_x = \frac{f}{P}\frac{\partial \alpha}{\partial x}\left[U\frac{\partial}{\partial \alpha}\log(S\bar{S}) - 2\frac{\partial U}{\partial \alpha}\right] + \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{U^2}{S\bar{S}}\right) \ldots (16).$$

Now differentiate the identity (11) with regard to α, keeping x, y, z, t constant, we obtain

$$\frac{\partial}{\partial \alpha}(S\bar{S}) - \frac{\partial}{\partial \alpha}(U\bar{U}) = 2kP.$$

Substituting in (16), making use of the relation $U\bar{U} = S\bar{S}$, we find that

$$2a_x = \frac{f}{P}\frac{\partial \alpha}{\partial x}\left[U\frac{\partial}{\partial \alpha}\log\left(\frac{\bar{U}}{U}\right) + 2\frac{kP}{\bar{U}}\right] - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\bar{U}}{U}\right)$$

$$= \frac{f}{P}\frac{\partial \alpha}{\partial x}\left[U\frac{\partial}{\partial \alpha}\log\left(\frac{\mu\bar{U}}{\lambda U}\right) + \frac{2kP}{\bar{U}}\right] - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\bar{U}}{\lambda U}\right)$$

$$= \frac{fU}{\mu P\bar{U}}\frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha}(\mu\bar{U}) - \frac{f}{\lambda P}\frac{\partial \alpha}{\partial x}\frac{\partial}{\partial \alpha}(\lambda U) + \frac{2kf}{\bar{U}} - \frac{Uf}{P}\frac{\partial}{\partial x}\log\left(\frac{\mu\bar{U}}{\lambda U}\right).$$