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**J. W. L. GLAISHER**

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ERRATUM—p. 46, last line, *cmr* " $= 1$ ."

# MESSENGER OF MATHEMATICS.

## ON CERTAIN SOLUTIONS OF MAXWELL'S EQUATIONS.

By *H. Bateman.*

### *Vector-fields with moving singular curves.*

§ 1. I HAVE shown elsewhere\* that it is possible to obtain a solution of Maxwell's equations which represents a vector field in which the electric and magnetic intensities are infinite at a moving point  $Q$ , whose coordinates at time  $\alpha$  are  $\xi, \eta, \zeta$ , and also along a moving curve attached to this point; the curve being the locus of a series of points projected from the different positions of  $Q$ , and travelling along straight lines with the velocity of light. The direction of projection for any position of  $Q$  was chosen so that it made an angle  $\theta$  with the tangent to the path of  $Q$  such that  $c \cos \theta = v$ , where  $v$  is the velocity of  $Q$ , and  $c$  the velocity of light. This condition is, however, not invariant under the transformations of the theory of relativity, and I now find that it is not necessary to restrict the direction of projection in the way described; the introduction of the restriction was due to the mistaken idea that the second of equations (291)† is a consequence of the first.

Let  $\alpha$  and  $\beta$  be defined as before by the equations

$$(x - \xi)^2 + (y - \eta)^2 + (z - \zeta)^2 = c^2(t - \tau)^2 \dots\dots(1),$$

$$l(x - \xi) + m(y - \eta) + n(z - \zeta) = c^2\rho(t - \tau) \dots\dots(2),$$

\* *The Mathematical Analysis of Electrical and Optical Wave Motion on the basis of Maxwell's Equations*, p. 128. This will be cited later as E.

† E, p. 129. The error in the proof occurs when the axis of  $y$  is chosen so that  $n_1 = 0$ ; this introduces a restriction, since  $l, m, n$  are generally complex. The same error occurs in one of my previous papers. *Annals of Mathematics* (1914).



2 *Mr. Bateman, Certain solutions of Maxwell's equations.*

where  $\xi, \eta, \zeta, \tau$  are functions of  $\alpha$  only, and  $l, m, n, p$  are functions of  $\alpha$  and  $\beta$  which depend linearly on  $\beta$ , so that

$$l = \beta l_1 - l_0, \quad m = \beta m_1 - m_0, \quad n = \beta n_1 - n_0, \quad p = \beta p_1 - p_0 \dots (3),$$

To make the values of  $\alpha$  and  $\beta$  unique, we write  $\tau = \alpha$  and introduce the inequality  $\tau \leq t$ . The quantities  $l, m, n, p$  must, moreover, be chosen so that  $l^2 + m^2 + n^2 = c^2 p^2$ , and so we have the relations

$$l_1^2 + m_1^2 + n_1^2 = c^2 p_1^2, \quad l_0 l_1 + m_0 m_1 + n_0 n_1 = c^2 p_0 p_1, \quad l_0^2 + m_0^2 + n_0^2 = c^2 p_0^2 \dots (4).$$

We now use the symbol  $f$  to denote an arbitrary function of  $\alpha$  and  $\beta$ , and write

$$P = \xi'(x - \xi) + \eta'(y - \eta) + \zeta'(z - \zeta) - c^2(t - \alpha) \dots (5).$$

The vector field which will be the subject of discussion is that defined by the electro-magnetic potentials

$$A_x = \frac{lf}{P}, \quad A_y = \frac{mf}{P}, \quad A_z = \frac{nf}{P}, \quad \Phi = \frac{c^2 pf}{P} \dots (6).$$

These have been shown to be wave-functions which satisfy the relation

$$\text{div } A + \frac{1}{c} \frac{\partial \Phi}{\partial t} = 0.$$

On calculating the components of the electric and magnetic intensities with the aid of the relations

$$H = \text{rot } A, \quad E = -\frac{1}{c} \frac{\partial A}{\partial t} - \frac{\partial \Phi}{\partial x},$$

we find, as before, that the component of the electric intensity along the radius from  $\xi, \eta, \zeta, \alpha$  to  $x, y, z, t$  is

$$-\frac{f}{P^2} [c^2 p - l\xi' - m\eta' - n\zeta'].$$

To make the electric charge associated with the singularity  $\xi, \eta, \zeta, \alpha$  a constant quantity  $4\pi$ , we choose  $f$  so that

$$f(c^2 p - l\xi' - m\eta' - n\zeta') = \xi'^2 + \eta'^2 + \zeta'^2 - c^2.$$

It will be convenient, however, to have a value of  $f$  independent of  $\beta$ , and so we shall introduce the condition

$$c^2 p_1 - l_1 \xi' - m_1 \eta' - n_1 \zeta' = 0;$$

as before, the value of  $f$  is then given by the equation\*

$$f(c^2 p_0 - l_0 \xi' - m_0 \eta' - n_0 \zeta') = c^2 - \xi'^2 - \eta'^2 - \zeta'^2 \dots (7).$$

We shall suppose that  $l_0, m_0, n_0, p_0$  are real, then  $f$  is real and  $l_1, m_1, n_1, p_1$  are generally complex quantities. Let  $\bar{l}_1, \bar{m}_1, \bar{n}_1, \bar{p}_1$  be the conjugate complex quantities, then we have the relations

$$\left. \begin{aligned} \bar{l}_1^2 + \bar{m}_1^2 + \bar{n}_1^2 &= c^2 \bar{p}_1^2, & l_0 \bar{l}_1 + m_0 \bar{m}_1 + n_0 \bar{n}_1 &= c^2 p_0 \bar{p}_1 \\ c^2 \bar{p}_1 - \bar{l}_1 \xi' - \bar{m}_1 \eta' - \bar{n}_1 \zeta' &= 0 \end{aligned} \right\} \dots (8).$$

Now let a set of real quantities  $\bar{l}_0, \bar{m}_0, \bar{n}_0, \bar{p}_0$  be chosen so that

$$\left. \begin{aligned} \bar{l}_0 \bar{l}_1 + \bar{m}_0 \bar{m}_1 + \bar{n}_0 \bar{n}_1 &= c^2 \bar{p}_0 \bar{p}_1, & \bar{l}_0 \bar{l}_1 + \bar{m}_0 \bar{m}_1 + \bar{n}_0 \bar{n}_1 &= c^2 \bar{p}_0 \bar{p}_1 \\ \bar{l}_0^2 + \bar{m}_0^2 + \bar{n}_0^2 &= c^2 \bar{p}_0^2, & \bar{l}_0 \bar{l}_1 + \bar{m}_0 \bar{m}_1 + \bar{n}_0 \bar{n}_1 - c^2 \bar{p}_0 \bar{p}_1 &= h \neq 0 \end{aligned} \right\} \dots (9).$$

Then if we write  $x - \xi = X, y - \eta = Y, z - \zeta = Z, t - \alpha = T,$

$$\left. \begin{aligned} c^2 p_1 T - l_1 X - m_1 Y - n_1 Z &= S, & c^2 \bar{p}_1 T - \bar{l}_1 X - \bar{m}_1 Y - \bar{n}_1 Z &= \bar{S} \\ c^2 p_0 T - l_0 X - m_0 Y - n_0 Z &= U, & c^2 \bar{p}_0 T - \bar{l}_0 X - \bar{m}_0 Y - \bar{n}_0 Z &= \bar{U} \end{aligned} \right\} \dots (10).$$

we find that, if  $h$  is suitably chosen, there is an identity of the type

$$S\bar{S} - U\bar{U} \equiv k(c^2 T^2 - X^2 - Y^2 - Z^2) = 0 \dots (11),$$

where  $k$  is a function of  $\alpha$  whose value may be determined by replacing  $T$  by 1,  $X$  by  $\xi'$ ,  $Y$  by  $\eta'$ , and  $Z$  by  $\zeta'$  in the identity. We thus find that

$$fk = \bar{l}_0 \xi' + \bar{m}_0 \eta' + \bar{n}_0 \zeta' - c^2 \bar{p}_0 \dots (12).$$

By considering the relations satisfied by  $l_1, m_1, n_1, p_1$ , we see that

$$\xi' = \lambda l_0 + \mu \bar{l}_0, \eta' = \lambda m_0 + \mu \bar{m}_0, \zeta' = \lambda n_0 + \mu \bar{n}_0, 1 = \lambda p_0 + \mu \bar{p}_0 \dots (13),$$

where  $\lambda$  and  $\mu$  are quantities to be determined. We deduce at once from these equations that

$$\begin{aligned} l_0 \xi' + m_0 \eta' + n_0 \zeta' - c^2 p_0 &= \mu h, & \bar{l}_0 \xi' + \bar{m}_0 \eta' + \bar{n}_0 \zeta' - c^2 \bar{p}_0 &= \lambda h, \\ \xi'^2 + \eta'^2 + \zeta'^2 - c^2 &= 2\lambda\mu h. \end{aligned}$$

\* It should be noticed that when this condition is satisfied the field specified by potentials of type  $A_x = \frac{Y'}{P} + \frac{Z'}{P}$  is conjugate to the field specified by Liénard's potentials of type  $A_x = \frac{Z'}{P}$  and the relation  $A_x^2 + A_y^2 + A_z^2 + A_0^2 - \Phi^2 = 0$  is satisfied. Compare this with the remark B, p. 135.

Hence 
$$2\lambda = \frac{\xi'^2 + \eta'^2 + \zeta'^2 - c^2}{l_0 \xi' + m_0 \eta' + n_0 \zeta' - c^2 p_0} = f \dots \dots \dots (14).$$

The expression for  $A_x$  can now be thrown into a more convenient form. Differentiating equation (2) we obtain

$$l = \left( \beta \frac{\partial S}{\partial \alpha} - \frac{\partial U}{\partial \alpha} \right) \frac{\partial \alpha}{\partial x} + S \frac{\partial S}{\partial x};$$

also  $\beta S = U$ , hence we may write

$$A_x = \frac{f}{P} \frac{\partial \alpha}{\partial x} \left[ \frac{U}{S} \frac{\partial S}{\partial \alpha} - \frac{\partial U}{\partial \alpha} \right] + \frac{Uf}{P} \frac{\partial}{\partial x} \log \left( \frac{U}{S} \right) \dots (15).$$

Now let  $\bar{A}_x$  be the complex quantity conjugate to  $A_x$ , then

$$\bar{A}_x = \frac{f}{P} \frac{\partial \alpha}{\partial x} \left[ \frac{U}{\bar{S}} \frac{\partial \bar{S}}{\partial \alpha} - \frac{\partial U}{\partial \alpha} \right] + \frac{Uf}{P} \frac{\partial}{\partial x} \log \left( \frac{U}{\bar{S}} \right).$$

and so if  $2a_x = A_x + \bar{A}_x$ , we have

$$2a_x = \frac{f}{P} \frac{\partial \alpha}{\partial x} \left[ U \frac{\partial}{\partial \alpha} \log(SS\bar{S}) - 2 \frac{\partial U}{\partial \alpha} \right] + \frac{Uf}{P} \frac{\partial}{\partial x} \log \left( \frac{U^2}{SS} \right) \dots (16).$$

Now differentiate the identity (11) with regard to  $\alpha$ , keeping  $x, y, z, t$  constant, we obtain

$$\frac{\partial}{\partial \alpha} (SS\bar{S}) - \frac{\partial}{\partial \alpha} (UU\bar{U}) = 2kP.$$

Substituting in (16), making use of the relation  $UU\bar{U} = SS\bar{S}$ , we find that

$$\begin{aligned} 2a_x &= \frac{f}{P} \frac{\partial \alpha}{\partial x} \left[ U \frac{\partial}{\partial \alpha} \log \left( \frac{\bar{U}}{U} \right) + 2 \frac{kP}{U} \right] - \frac{Uf}{P} \frac{\partial}{\partial x} \log \left( \frac{\bar{U}}{U} \right) \\ &= \frac{f}{P} \frac{\partial \alpha}{\partial x} \left[ U \frac{\partial}{\partial \alpha} \log \left( \frac{\mu \bar{U}}{\lambda U} \right) + \frac{2kP}{U} \right] - \frac{Uf}{P} \frac{\partial}{\partial x} \log \left( \frac{\mu \bar{U}}{\lambda U} \right) \\ &= \frac{fU}{\mu P U} \frac{\partial \alpha}{\partial x} \frac{\partial}{\partial \alpha} (\mu \bar{U}) - \frac{f}{\lambda P} \frac{\partial \alpha}{\partial x} \frac{\partial}{\partial \alpha} (\lambda U) + \frac{2kf}{U} - \frac{Uf}{P} \frac{\partial}{\partial x} \log \left( \frac{\mu \bar{U}}{\lambda U} \right). \end{aligned}$$