ON THE CARDIOIDS FULFILLING CERTAIN ASSIGNED CONDITIONS: A DISSERTATION

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On the Cardioids Fulfilling Certain Assigned Conditions: A dissertation by Mary Kelley

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ON THE CARDIOIDS FULFILLING CERTAIN ASSIGNED CONDITIONS: A DISSERTATION

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On the Cardioids
fulfilling
Certain Assigned Conditions

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A DISSERTATION

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Introductory

The tricuspidal, bicircular quartic of the third class defined by the

$$\left.\begin{array}{l}\text{Cartesian equation } (x^2+y^2+ax)^2=a^2(x^2+y^2)\\ \text{polar equation } \rho=a(1-\cos\theta)\end{array}\right\},$$

and commonly known as the Cardioid, has for many years been the object of mathematical investigation. It has lately been studied by Raymond Clare Archibald in his Inaugural-Dissertation "The Cardioide and Some of Its Related Curves" (Strassburg, 1900), which work contains an historical sketch of the curve and a presentation of results prior to the year of its publication. Since then, the only work on the subject of considerable length is Professor Archibald's paper, "The Cardioid and Tricuspid: Quartics with Three Cusps."* Besides this there have appeared a few detached problems† and contributions in periodicals treating the curve from either a metric or a projective standpoint.

The chief characteristic of former research along this line seems to be the examination of the cardioid as a fixed curve and the consideration of its properties as such. The present investigation starts from a different point of view, which we may outline as follows:

In general, a curve of the fourth degree is capable of satisfying 14 conditions. The cardioid, however, having 3 cusps, two of which are at the fixed (circular) points I and J, can be subjected to only 4 conditions.‡ If, then, 3 conditions be imposed, there are ∞^1 curves satisfying them; therefore, the special elements (cusp, focus, double tangent) describe definite loci. If 4 conditions are given, there are a finite number of curves satisfying them. It is our purpose to obtain the loci generated in the first case; and, in the second, to determine the number and (where possible) the reality of cardioids for various kinds of assigned conditions.

The co-ordinate system which lends itself most readily to such an investigation is the system of conjugate (also called circular) co-ordinates, in which a point is named by a vector which has its

* *Annals of Mathematics*, 2d S. 4, 1902-3, pp. 95ff.

† Cf., for example, Questions 14,435; 16,203; 16,394. *Ed. Times* (London).

‡ Cusps at I and J count for 4 conditions each. The third cusp counts for 2 more conditions. Thus, the 14 conditions reduce to $14-(2\times4+2)=4$.

initial extremity at some fixed point, called the origin. Denoting a vector by z, its projections on the real and the imaginary axes are designated by x, y, respectively; and

$$z = x + iy.^*$$

With a vector z is associated its conjugate z'. This may be geometrically defined as its reflection in the real axis. Accordingly,

$$z' = x - iy.$$

All vectors may be considered as obtained from the standard unit vector by means of a stretch and a turn. Turns are regularly designated by the letter t; fixed turns, by t_1, t_2, t_3 . . . ; variable turns, by t, t^1, τ . . . .†

Regarding the cardioid as the epicycloid generated by equal circles, the map-equation of the curve is

$$z = r(2t - t^2).$$

The centre of the fixed circle is at O, which point, we shall, with Professor Morley,‡ call the centre of the cardioid. This point is also the singular focus of the curve. The cusp is at $z = r$ (r being on the real axis).

This map-equation involves its conjugate,

$$z' = r\left(\frac{2}{t} - \frac{1}{t^2}\right).$$

These two equations taken together may be considered the parametric equations of the curve.

A cardioid with centre at z_0 and any orientation has for its map-equation:

$$z - z_0 = 2at - a't^2$$

$$z = z_0 + \frac{a^2}{a'} \text{ gives the cusp.}$$

The angle made by the axis of the curve with the axis of reals is θ, where

$$e^{2i\theta} = \left(\frac{a}{a'}\right)^3.$$

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* The symbol i denotes, as usual, $\sqrt{-1}$.

† Cf. Morley, "On Reflexive Geometry," *Transactions of the American Mathematical Society*, 1907, Vol. 8, p. 14.

‡ "Metric Geometry of the Plane N-Line," *Transactions of the American Mathematical Society*, 1900, Vol. 1, p. 105.

Any tangent is given by

$$(z-z_0)-(z'-z'_0)t^3-3at+3a't^2=0;$$

the double tangent, by

$$a'^3(z-z_0)+a^3(z'-z'_0)=3a^2a'^2.$$

The kinds of data we shall consider are: point, line, centre, cusp, or double tangent given, to solve a specified problem. The last three are equivalent to 2 conditions each. The problems arising naturally fall into 2 classes:

I. Given 3 conditions, find specified loci; as,

(1) Given the centre and a line; find the locus of cusps.
(2) Given the centre and a point; find the locus of cusps.
(3) Given the cusp and a line; find the locus of centres.
(4) Given the cusp and a point; find the locus of centres.
(5) Given the double tangent and a line; find the locus of cusps.
Given the double tangent and a line; find the locus of centres.
(6) Given the double tangent and a point; find the locus of cusps.
Given the double tangent and a point; find the locus of centres.
(7) Given 3 lines, find the locus of centres.
(8) Given 2 lines and 1 point, find the locus of centres.
(9) Given 1 line and 2 points, find the locus of centres.
(10) Given 3 points, find the locus of cusps.

II. Given 4 conditions, find how many solutions there are; as,

(a) Given the centre and 2 lines, how many cardioids are there? How many are real?
(b) Given the centre, a point and a line. Apply the same questions.
(c) Given the centre and 2 points.
(d) Given the cusp and 2 lines.
(e) Given the cusp, 1 line and 1 point.
(f) Given the cusp and 2 points.
(g) Given the double tangent and 2 lines.
(h) Given the double tangent, 1 line and 1 point.
(i) Given the double tangent and 2 points.
(j) Given 4 lines.
(k) Given 3 lines and 1 point.
(l) Given 2 lines and 2 points.

(m) Given 1 line and 3 points.
(n) Given 4 points.

Again, there is an evident division of the problems into simpler and more difficult cases. It is not the intrinsic value of some of the simple cases which authorizes their appearance in this work, but rather their importance in the solution of some of the more difficult problems.

Before taking up the problems, it will be well to indicate a guiding principle which will be found of great importance for cases where the cardioids are to touch several lines. It may be stated thus:

If θ be the angle between 2 lines which touch a cardioid, the angle between the cuspidal rays to the points of tangency has some one of the values $\frac{2}{3}\theta$, $\frac{2}{3}(\theta+360°)$, $\frac{2}{3}(\theta+720°)$.

For the case of parallel tangents, this specializes to the well-known theorem:

The points of contact of any 3 parallel tangents to a cardioid subtend angles of 120° at the cusp.

With these considerations premised, we shall proceed to a treatment of the simpler cases.