TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, VOL. 6, NO. 3, PP. 353-415: THE RELATION OF THE PRINCIPLES OF LOGIC TO THE FOUNDATIONS OF GEOMETY

Published @ 2017 Trieste Publishing Pty Ltd

ISBN 9780649339037

Transactions of the American Mathematical Society, Vol. 6, No. 3, pp. 353-415: The Relation of the Principles of Logic to the Foundations of Geomety by Josiah Royce

Edited by Trieste Publishing Pty Ltd.
Cover @ 2017

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Reprinted from the
TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY
Vol. 6, No. 3, pp. 353-415
July, 1905

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INTRODUCTION.

In the year 1886, in the Philosophical Transactions of the Royal Society, Mr. A. B. KEMPE published *A Memoir on the Theory of Mathematical Form*, in which, amongst other matters, he discussed the fundamental conceptions both of symbolic logic and of geometry. The ideas there indicated were further developed, by Mr. KEMPE, in an extended paper *On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points*, in the Proceedings of the London Mathematical Society for 1890. Despite the close attention that has since then been devoted to the study of the foundations of geometry, Mr. KEMPE's views have remained almost unnoticed. They concern, however, certain matters which recent research enables us to regard with increasing interest. I have been led, therefore, to attempt a restatement of KEMPE's logical-geometrical theory. The restatement has led me to conceptions which, although implied in those which Mr. KEMPE emphasizes, present a number of aspects which I believe to be novel, so that a considerable part of the present research follows a path of its own. My introductory words will indicate the nature of KEMPE's contribution to the problem of the foundations of geometry, the kind of task which his work has set before me, and my own main interest in preparing this paper.

The fundamental ordinal relation of geometry is the relation which can be, at pleasure, described as the triadic relation "between," or as an asymmetrical, transitive dyadic relation, such as "before," or "antecedent to," or "sequent to." Essentially the same relation is at the root of all serial order, and on this basis the logic of such order has lately been elaborately discussed by Mr. BERTRAND RUSSELL, in his *Principles of Mathematics*.

The axioms of geometry, as Dr. VEBLEN has stated them (Transactions of the American Mathematical Society, July, 1904), consist (1) of

* Presented to the Society April 29, 1905. Received for publication May 5, 1905.

assertions characterizing the "between" relation, and duly restricting the application of this relation so far as the "lines" of geometry are concerned, and (2) of existential propositions defining certain entities that shall possess the relation. A similar prominence of asymmetrical transitive relations appears in Dr. HUNTINGTON'S various *Sets of Postulates* for numbers, groups, etc. (Ibid., January and April, 1905).

The algebra of logic may be viewed (as Dr. HUNTINGTON, following Mr. PEIRCE and SCHROEDER, has lately afresh shown in detail), as depending upon the relation of inclusion or subsumption, sometimes symbolized by $\prec$. This relation is dyadic and transitive, and may be either symmetrical or unsymmetrical. Upon the basis of this one relation we can define the various operations of formal logic, such as logical multiplication and addition. If the relation $\prec$ is in a given instance symmetrical, it ensures what is commonly viewed as the "uniqueness" of an entity. That is: $a \prec a$; and if $a \prec b$, while $b \prec a$, than $b = a$ (see Dr. HUNTINGTON'S paper of July, 1904, in these Transactions, for a fuller statement of the various results of these considerations). The relation $a \prec b$, in so far as it obtains between non-equivalent elements, may serve to define linear series: $a \prec b \prec c \prec d$, etc.; where $a \prec c$, and $a \prec d$. In such a series c may obviously be said to lie "between" b and d, and the analogy to the geometrical relation "between" is in so far plain. "Dense," and in fact, continuous linear series of the subsumption type can be conceived after the analogy of point series. But on the other hand, a system of logical classes differs, with respect to linear relations, from a system of points on a line in *two* very notable ways: —

(1) If $a \prec b \prec d$, and if it is also true that $a \prec c \prec d$, any one of the three relations $b \prec c$, $c \prec b$, $c = b$, is indeed *possible*; but, in case of the logical entities, it is also possible that b and c are such that no one of these relations actually holds between these two. Thus, Siberia is included within the Russian Empire, which itself may be viewed as included within the "Eurasian" continent. And Siberia is also included in Asia, which may also be regarded as included within the "Eurasian" continent. These subsumptions are transitive, and in so far linear in their type. Yet the Russian Empire and Asia do not form a pair possessing the relation $\prec$, read in either sense.

(2) If $a \prec b \prec c \prec d$, and if, also, $i \prec b \prec c \prec j$, the relations of i and a, of j and d are similarly left indeterminate. These relations need not be directly expressible in terms of $\prec$ at all. That is, nothing in the logical relations forbids linear series (whether dense, or continuous or not) to have two or more "points," i. e., elements, in common, while any number of the other elements of the series remain entirely distinct. The logical lines, as Mr. KEMPE observes, may intersect any number of times.

For this very reason, however, the system of logical entities may be viewed simply as much more general and inclusive than the system of the points of space. And thus it becomes possible to regard a given space-form as *a selection from amongst the entities present in a system that exemplifies the logical relation* $\prec$. That is: One may view the points of a space as a select set of logical elements, chosen, for instance, from a given "universe of discourse." This thought, whose possible fruitfulness for the logical development of the foundations of geometry I regard as highly notable, is the essential thought at the basis of Mr. KEMPE's paper of 1890, cited at the outset of this introduction.

The reason why such a thought seems promising is this: The relations amongst logical entities are, in any case, the most fundamental relations that we know. Experience shows us in the outer world those ordinal space-relations which geometry generalizes in the concept of "between." But our own thinking processes show us the meaning of the logical relation $\prec$. The latter relation, then, is more suited to be the basis for a theory of the logic of an exact science, in case we can only so define and restrict its application that our ideal geometrical relations can come to be viewed as special instances of those forms which we can develop by the use of pure logic.

Mr. KEMPE's procedure, in the paper of 1890, is, in bare outline, as follows: He sets out, not by assuming the ordinary algebra of logic, but by defining, through postulates, a purely abstract set of entities called by him the "base-system," and a relation which may be viewed as a generalized "between." The latter is the relation which, in its most general form, is characteristic of what KEMPE himself calls, later in his paper, "flat collections" of any number the elements of the "base-system." But the relation first appears as a *triadic* relation, and is so characterized in the postulates. KEMPE uses the notation: $ac \cdot b$, to mean the assertion: "a, b, and c, form a 'linear triad,' with b between a and c." So far the expressions used resemble those for Dr. VEBLEN's generalized relation "in the order abc." But KEMPE's linear triad has these fundamental properties: (1) "If $ab \cdot c$, and $a = b$, then $c = a = b$." (2) "If $a = b$, then $ac \cdot b$ and $bc \cdot a$, whatever entity of the system c may be."* In other words, KEMPE permits the "between" relation to hold where the related elements are, for all the purposes of the operations of the system, identical; and then he defines the distinctness of elements by means of a restriction of the relations that are permitted to hold in triads of distinct elements. The result is that the "between" relation becomes Dr. VEBLEN's "in the order," whenever the elements are all distinct.

The other properties of the "between" relation which are in question for

*I vary a little the order of Mr. KEMPE's statement of his principles. The relation $=$ is defined by Mr. KEMPE only in a very highly abstract form which I need not here attempt to discuss. Geometrically interpreted, if this relation holds between points, these become identical points.

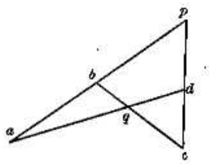
Mr. KEMPE, are obtained by him through assuming two forms of triadic "transversal" propositions as fundamental postulates, viz.: *

I. If two linear triads, $ap \cdot b$ and $cp \cdot d$, exist, such that (as indicated by the notation), b is between a and p, and d is between c and p, then there exists an entity, q, which lies, in a linear triad, between a and d, and, in another linear triad, between b and c.

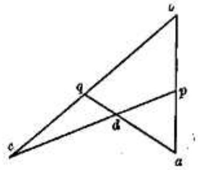
II. If, in the linear triads $ab \cdot p$ and $cp \cdot d$ (as indicated by the notation), p lies between a and b, and d between c and p, then q exists such that q lies, in a linear triad, so that d is between a and q, while, in another linear triad, q is between b and c.

If one interprets these assertions as relating to points in space, they become assertions obviously relating, respectively, to the diagrams following. But, as

I.



II.



they are stated at the outset of Mr. KEMPE's paper, these principles have no specification beyond what the general properties of the linear triad, as just defined, predetermine.

One other existential proposition Mr. KEMPE uses as his *fifth* fundamental principle. This is simply the proposition that any entity belongs to the base-system whose presence there is not inconsistent with the four other principles, — a proposition which of course formally renders the two existential principles, here numbered I and II, superfluous; and which leaves the account of the "base system" inevitably somewhat unsatisfactory.

Mr. KEMPE now proceeds upon this basis, to show, by a decidedly original, although necessarily intricate procedure, that the elements of the base system, as thus defined, possess the properties and relations of a system of logical classes, or of other entities subject to the algebra of logic. In other words, he develops the entire algebra of logic, including the definitions and properties of the operations of logical multiplication and logical addition, without any other assumptions than those simple properties of the "between" relation which have just been stated. The proofs given are such as to apply to any finite number of the elements. Mr. KEMPE leaves, however, some doubt as to infinite collections.

* I vary slightly Mr. Kempe's mode of ennunciating these existential propositions at the outset of his paper.

Highly instructive observations are incidental to this development. The system of logical entities appears as possessing a thoroughly symmetrical structure. The "zero"-element and the "universe"-element have no essential distinction from any other similarly related pair of "obverse" elements. Negatives, in general appear as "obverses," because of the symmetrical contrast of their respective relations to the remainder of the system. All the fundamental relations of logic appear as triadic rather than as dyadic. But upon this triadic basis, polyadic relations also develop — the relations of KEMPE's "Flat-collections." These collections, thus named by reason of their resemblance to the various possible configurations of points in an n-dimensional space — "on a line," "on a plane," "in a three dimensional space," etc. — are Mr. KEMPE's means of relating the purely logical to the geometrical entities.

The junction of his principles with the regular algebra of logic once completed, although leaving certain doubts as to the application of his proof to infinite sets, KEMPE proceeds to the geometrical application. By (1) selecting certain "linear sets" of the elements of the base system; and (2) selecting from these sets those which conform to a new principle (here for the first time introduced into the essay), namely, to the principle that any two of the elements of a selected linear set shall determine the whole linear set to be selected, KEMPE is in possession of a system of foundations for a geometry of a "flat space" of n dimensions. The further development of such a geometry is indeed merely sketched in the paper in question. But since the "triangle-transversal" axiom is provided for by the initial principles of the system, and since, by the selection of the linear sets of elements, the ordinary properties of the geometrical "between," and the axiom as to the determination of a line by any two of its points are now also secure, KEMPE's result, although only indicated in his text, is in the main clear. *A space of n dimensions is a select class or set of elements which themselves are entities in a logical "field."* The selection of the entities of a given space is arbitrary; and so the space-forms whose entities are selected may be varied in any way whatever which is consistent with the triangle-transversal-axiom, and with the properties of the generalized between-relation. The problem of the continuity of the geometric sets is only very generally treated, and is not solved.

The wide outlook thus suggested into the theory of space-forms certainly deserves to be better considered than KEMPE's treatment of the subject has so far been considered by mathematicians. For me, however, as a student of philosophy, a still further interest attaches to those results which I have thus suggested, an interest which my mathematical colleagues may also share.

The problem of the foundations of geometry is only a part of that general problem regarding the fundamental concepts of the exact sciences which is now so widely studied. KEMPE's research suggests that, since metrical relations, and therefore (as KEMPE himself briefly indicated in his *Theory of Mathematical Form*, § 309), the whole algebra of ordinary quantity, can be reduced, in any system of three or more dimensions, to a series of propositions based upon purely ordinal relations, — the entire system of the relationships of the exact sciences stands *in a much closer connection with the simple principles of symbolic logic* than has thus far been generally recognized.

Mr. BERTRAND RUSSELL, using very different methods, insists, indeed, in general, upon the closeness of such a connection. But the distinction between the "logic of relations," and the older "logic of classes," and of "propositions," — a distinction which Mr. BERTRAND RUSSELL in his *Principles of Mathematics* regards as something quite fundamental, seems to me to become, in the light of KEMPE's research, a distinction probably quite superficial. Hence to my mind, Mr. KEMPE's theory goes far deeper than Mr. RUSSELL's. Give us a system of entities of the types of logical classes, and we shall find that their relations (all statable in terms of KEMPE's "between"), are already (quite apart from a separate "logic of relations"), certainly as rich as the totality of the relations studied in geometry, and are, for reasons upon which KEMPE has dwelt, probably as rich as the totality of the relations known to the exact sciences, at least so far as the latter have yet been developed. The bare prospect of such a result deserves a careful consideration, in case one takes interest in the unification of the categories of science. KEMPE's theory promises such an unification.

The present memoir proposes to contribute towards a more precise statement of the theory thus outlined. At the basis of my own discussion, I place, however not KEMPE's "between" relation, but another fundamental relation of symbolic logic which has the interest of being absolutely *symmetrical*, while, when it obtains amongst n entities, it permits (upon the basis of certain simple existential propositions), the definition of the properties of KEMPE's "flat collections," and so the definition of asymmetrical relations of a very high degree of complexity. This change of starting point is the prime novelty of the present discussion, as contrasted with KEMPE's.

The *contrast between symmetrical and unsymmetrical relations* seems, to the ordinary view, absolute. Mr. RUSSELL, in his late volume, so treats it. Geometry, and the ordinary algebra of quantity (as these subjects are usually treated), seem to depend on regarding the distinction as quite fundamental. In symbolic logic, however, as Mrs. LADD-FRANKLIN long ago pointed out (in her paper on the algebra of logic in the volume called "*Studies in Logic by members of the Johns Hopkins University*," Boston, 1883), a "symmetrical copula," namely that of "inconsistency," or of "opposition," can be made to accomplish all the work of the ordinary unsymmetrical copula $\prec$. In other words, if I have otherwise defined the meaning of "not," the statement "x is inconsistent with not-y," means the same as "x implies y." The copula in the former case is