SOLUTIONS OF THE EXAMPLES IN A SEQUEL TO ELEMENTARY GEOMETRY

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SOLUTIONS OF THE EXAMPLES IN A SEQUEL TO ELEMENTARY GEOMETRY

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NOTICE TO THE READER

THE solutions given in this *Key* are not always applicable, without slight modification, to every possible figure which may be drawn to illustrate the problem concerned. Any reader who from this cause, or any other, finds serious difficulty in the use of this work, is invited to communicate with the author,

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ABBREVIATIONS

| = straight line ; / = divided by.

⊥ = perpendicular, is perpendicular to, orthogonal, is orthogonal to ; $\perp^{ly}$ = perpendicularly.

∥ = parallel, is parallel to ; $\parallel^{m}$ = parallelogram.

⊙ = circle ; $\odot^{r}$ = circular ; ⊙ of s. = circle of similitude.

c. of g. = centre of gravity ; c. of i. rotation = centre of instantaneous rotation ; c. of m. = centre of mass ; c. of s. = centre of similitude ; const. = constant.

h. c. = homothetic centre ; h. r. = homothetic ratio ; h^{c} = harmonic ; h^{ly} = harmonically.

int^{n} = intersection.

max. = maximum ; min. = minimum.

N.P.C. = nine-point circle.

o. c. = orthocentre.

pt. = point ; $proj^{n}$ = projection.

quad. = a figure with four sides and four angles.

r. a. = radical axis ; r. c. = radical centre.

ult^{ly} = ultimately.

vol. = volume.

w. r. to = with respect to.

KEY TO SEQUEL

CHAPTER I

Page 2. Ex. Let the ⊥ bisector meet AB at L and CD at M; then B, D are the reflexions of A, C in LM. Hence $\angle ACD = BDC = 180^\circ - ABD$ by $\|^s$.

Page 3. Ex. Since $AQ = \frac{1}{2}AP$, the locus of Q is homothetic to l, the locus of P; i.e. is a |, ∥ to l and half-way between A and l.

Page 5. Ex. 1. Let OP cut AB at P''. Then since $PB \parallel OA$, the Δ^s OAP'', PBP'' are similar, $\therefore AP'' : BP'' :: OA : PB :: AP' : BP'$ by hyp. Hence P'' coincides with P'; i.e. O, P, P' are collinear. Also $OP' : OP :: AP' : AB$, a const. ratio. Hence P and P' describe homothetic and, therefore, similar curves.

Ex. 2. Let OO' and PP' meet at S. Then $O'S : OS :: O'P' : OP = k$, say, $\therefore O'S = k . OS$. Hence S is a fixed pt. Also $SP' : SP :: O'P' : OP = k$. Hence $SP' = k . SP$.

Page 6. Ex. Let P' be the reflexion of P in OA and P'' of P' in OB. Then $\angle POA = P'OA$, $P''OB = P'OB$, $\therefore POP' + P'OP'' = 2AOP' + 2P'OB = 2 . 90^\circ = 180^\circ$. Also $PO = OP' = OP''$. Hence P, O, P'' are collinear and $PO = OP''$.

Page 8. Ex. We shall first prove that the Δ^s $P'CO$ and OAP are similar. Now $P'C : OA :: P'C : BC :: AB : AP$ (from similar Δ^s $P'CB$, BAP) $:: OC : AP$, $\therefore P'C : OC :: OA : AP$. Also the $\angle^s$ $P'CO$, OAP are equal; for $P'CO = P'CB + BCO = BAP + OAB = OAP$. Hence the Δ^s $P'CO$, OAP are similar, $\therefore OP' : OP :: OC : AP$, a const. ratio. Hence $OP' = k . OP$. Again $\angle POP'$ is const.; for $POP' = AOC - P'OC - AOP = AOC - P'OC - CP'O$ (for the similar Δ^s $P'CO$, OAP give $\angle CP'O = AOP$) $= BCX - P'CX$ (producing OC to X) $= BCP'$. Hence, since $OP' = k . OP$ and $\angle POP'$ is const., P and P' describe similar curves.

Page 12. Ex. 1. In the figure on p. 10, let ABC, $A'B'C'$ be two such Δ^s, X, Y, Z being the given pts. and OA, OB the given |s. Then since AA', BB', CC' concur, the locus of C' (taking ABC as fixed) is the fixed | OC.

Ex. 2. Since ABC, $A'B'C'$ are in perspective, AA', BB', CC' concur. Hence ABC', $A'B'C$ are copolar and ∴ coaxal. Hence $(BC' ; B'C)$, $(C'A ; CA')$, $(AB ; A'B')$ are collinear.

Ex. 3. Since AB, $A'B'$, $A''B''$ concur, $AA'A''$ and $BB'B''$ are copolar and ∴ coaxal; ∴ $(AA' ; BB')$, $(A'A'' ; B'B'')$, $(A''A ; B''B)$ are collinear, i.e. the centres of perspective are collinear.

Ex. 4. Consider the Δs whose sides are AB, $A'B'$, $A''B''$ and BC, $B'C'$, $B''C''$. Corresponding sides meet at B, B', B'' which are collinear. Hence the Δs are coaxal and ∴ copolar. Hence the |s joining $(AB ; A'B')$ to $(BC ; B'C')$, $(A'B' ; A''B'')$ to $(B'C' ; B''C'')$, $(A''B'' ; AB)$ to $(B''C'' ; BC)$ concur; i.e. the axes of perspective concur.

CHAPTER II

Page 14. Ex. 1. The Δs $OA'B$, $OA'C$ are congruent, ∴ $\angle BOL = COL$, ∴ $\angle BAL = CAL$. Again $\angle LAM = 90°$; ∴ AM is the external bisector of BAC since AL is the internal bisector.

Ex. 2. The Δs $BA'L$, $CA'L$ are congruent, ∴ $BL = CL$. Also $\angle PBL = QCL$ (since $BACL$ is cyclic) and $\angle BPL = CQL$ ($= 90°$). Hence the Δs LBP, LCQ are congruent.

Now $AP + PB = AB$. Again $AP - PB = AQ - CQ$ (from the congruent Δs APL, AQL) $= AC$. Hence $AP = \frac{1}{2}(AB + AC)$ and $PB = \frac{1}{2}(AB - AC)$.

Ex. 3. We have $a = 2R \sin A = 2R \sin A' = a'$, since $A = A'$; so $b = b'$, $c = c'$.

Ex. 4. For brevity take six vertices $A_1, A_2, A_3, A_4, A_5, A_6$. Let P be the point. Let $PA_1 = a_1, \ldots$ and let the ⊥s from P on A_1A_2, A_2A_3, ... be $p_1, p_2, \ldots$. Then, since $bc = 2pR$ in any Δ, we have

$$2Rp_1 = a_1a_2,\ 2Rp_2 = a_2a_3$$
$$2Rp_3 = a_3a_4,\ 2Rp_4 = a_4a_5$$
$$2Rp_5 = a_5a_6,\ 2Rp_6 = a_6a_1.$$

$$\therefore\ 8R^3 p_1 p_3 p_5 = a_1 a_2 a_3 a_4 a_5 a_6 = 8R^3 p_2 p_4 p_6;$$
$$\therefore\ p_1 p_3 p_5 = p_2 p_4 p_6.$$

The same proof holds if we take any even number of sides.

Ex. 5. For brevity take three vertices A_1, A_3, A_5. First take A_2, A_4, A_6 near to A_1, A_3, A_5 on the ⊙. Then, as above, $p_1 p_3 p_5 = p_2 p_4 p_6$. Now make A_2, A_4, A_6 coincide with A_1, A_3, A_5. Then p_1, the ⊥ on A_1A_2, becomes the ⊥ on the tangent at A_1; so p_3, p_5. A similar proof holds for any number of vertices.

Page 17. § 4. Ex. (i) $A'X_1 = A'B - BX_1 = \frac{1}{2}c - (s-c) = A'C - CX = A'X$.

(ii) $XX_2 = BX_2 - BX = s - (s-b) = b$.

Page 17. § 5. Ex. 1. From any pt. A on the first ⊙, c, draw the tangents to the second ⊙, i, cutting c at B and C. Let I', r' be the incentre and inradius of ABC. Then I, I' lie on the internal bisector of A. Let AI cut c at L. Now $R^2 - OI^2 = AI \cdot IL$ (as in the text) $= 2Rr$ (by hyp.); also $AI' \cdot I'L = 2Rr'$ (as in the text). Hence $IL : I'L = r | AI : r' | AI' = 1$ (by similar Δ^s got by dropping ⊥s from I, I' on AC). Hence I and I' coincide. Hence i is the incircle of ABC; for it has I as centre and touches AB.

Ex. 2. Let the ⊙ with centre L and radius LB or LI cut IL again at I'. Then $IBI' = 90°$. But $IBI_1 = 90°$. $\therefore$ I' and I_1 coincide, $\therefore$ $LB = LI_1$.

Again $OI_1^2 - R^2 = I_1L \cdot I_1A$ (as in the text) $= LB \cdot I_1A$. Also $2Rr_1 = LM \cdot I_1Z_1$. Hence we have to prove that $LB \cdot I_1A = LM \cdot I_1Z_1$, or $LB : LM :: I_1Z_1 : I_1A :: IZ : AI$. Now see the text.

Ex. 3. The square of the tangent $= I_1L \cdot I_1A = 2Rr_1$ by the above.

Ex. 4. If $R = 2r$, $R^2 = 2Rr$, $\therefore$ $OI^2 = 0$. Hence O and I coincide, at S, say. Then, since $IY = IZ$, $SY = SZ$; and $AS = AS$ and $Y = Z = 90°$. Hence $AY = AZ$. But since S is O, $AY = \frac{1}{2}AC$ and $AZ = \frac{1}{2}AB$. Hence $AC = AB$; so $AB = BC$.